CS 105

"Tour of the Black Holes of Computing!"

Floating Point

Topics

- IEEE Floating-Point Standard
- Rounding
- Floating-Point Operations
- Mathematical Properties

Floating-Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why it is not true, ideally with an example

```
int x = foo();
float f = bar();
double d = baz();
```

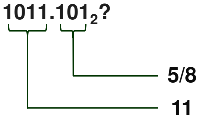
Assume neither `d` nor `f` is NaN

Assume a 32-bit machine

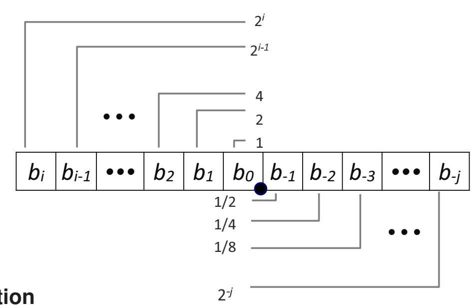
- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f)`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f > -d`
- `d * d >= 0.0`
- `(f+d)-d == f`

Fractional binary numbers

What is 1011.101_2?



Fractional Binary Numbers



Representation

- Bits to right of "binary point" represent fractional powers of 2
- Represents rational number: $\sum_{k=-j}^{i} b_k \times 2^k$

Fractional Binary Numbers: Examples

- **Value** — **Representation**
  - 5 3/4 — `101.11`$_2$
  - 2 7/8 — `10.111`$_2$
  - 1 7/16 — `1.0111`$_2$

- **Observations**
  - Divide by 2 by shifting right (unsigned)
  - Multiply by 2 by shifting left
  - Numbers of form 0.111111…$_2$ are just below 1.0
    - 1/2 + 1/4 + 1/8 + … + 1/2^i + … → 1.0
    - Use notation 1.0 – ε

Representable Numbers

Limitation #1

- Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

Value	Representation	Decimal Representation
1/3	`0.0101010101[01]…`$_2$	`0.333333333…`
1/5	`0.001100110011[0011]…`$_2$	`0.200000000…`
1/10	`0.0001100110011[0011]…`$_2$	`0.100000000…`

Limitation #2

- Just one setting of binary point within the *w* bits
  - Limited range of numbers (very small values? very large?)

IEEE Floating Point

IEEE Standard 754

- Established in 1985 as uniform standard for floating-point arithmetic
  - Before that, many idiosyncratic formats
- Supported by all major CPUs

Driven by numerical concerns

- Nice standards for rounding, overflow, underflow
- Hard to make go fast
  - Numerical analysts predominated over hardware types in defining standard
  - Nevertheless, talented engineers have succeeded

Floating-Point Representation

Numerical Form

- $-1^s\, M\, 2^E$
  - Sign bit *s* determines whether number is negative or positive (negative zero representable)
  - Significand *M* normally a fractional value in range [1.0, 2.0).
  - Exponent *E* weights value by a power of two

Encoding

- MSB is sign bit
- `exp` field encodes *E* (emphasis on "encodes")
- `frac` field encodes *M* (likewise)

Precision Options (Not to Scale)

Single precision: 32 bits float

s	exp	frac
1	8 bits	23 bits

Double precision: 64 bits

s	exp	frac
1	11 bits	52 bits

Extended precision: 80 bits (Intel only)

s	exp	frac
1	15 bits	63 or 64 bits

"Normalized" Values

$v = (-1)^s M\ 2^E$

When: exp ≠ 000…0 and exp ≠ 111…1

Exponent coded as a *biased* value: $E = Exp - Bias$

- *Exp*: unsigned value of exp field
- *Bias* = $2^{k-1} - 1$, where *k* is number of exponent bits
  - Single precision: 127 (Exp: 1…254, E: -126…127)
  - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

Significand coded with *implied* leading 1: $M = 1.xxx…x_2$

- xxx…x: bits of frac field
- Minimum when frac=000…0 (M = 1.0)
- Maximum when frac=111…1 (M = 2.0 – ε)
- Get extra leading bit for "free"

Normalized Encoding Example

Value

```
float f = 15213.0;
```

- $15213_{10} = 11101101101101_2 = 1.1101101101101_2 \times 2^{13}$

Significand

M = 1.1101101101101_2

frac= $11011011011010000000000_2$

Exponent

E = 13

Bias = 127

Exp = 140 = 10001100_2

Floating-Point Representation:

Hex:	4	6	6	D	B	4	0	0
Binary:	0100	0110	0110	1101	1011	0100	0000	0000

140: 100 0110 0

15213: 1110 1101 1011 01

Denormalized Values

$v = (-1)^s M\ 2^E$
$E = 1 - Bias$

Condition: exp = 000…0

Exponent value: $E = 1 - \text{Bias}$ (instead of $E = 0 - Bias$)

Significand coded with implied leading 0: $M = 0.xxx…x_2$

- xxx…x: bits of `frac`

Cases

- `exp` = 000…0, `frac` = 000…0
  - Represents zero value
  - Note distinct values: +0 and –0 (why?)
- `exp` = 000…0, `frac` ≠ 000…0
  - Numbers closest to 0.0
  - Equispaced

Special Values

Condition: `exp` = `111…1`

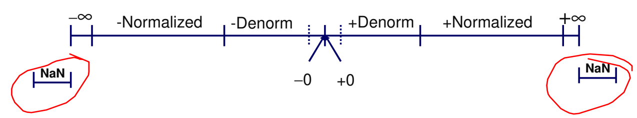
Case: `exp` = `111…1`, `frac` = `000…0`

- Represents value ∞ (infinity)
- Operation that overflows
- Both positive and negative
- E.g., 1.0/0.0 = –1.0/–0.0 = +∞, 1.0/–0.0 = –∞

Case: `exp` = `111…1`, `frac` ≠ `000…0`

- Not-a-Number (NaN)
- Represents case when no numeric value can be determined
- E.g., sqrt(–1), ∞ – ∞, ∞ × 0

Visualization: Floating-Point Encodings

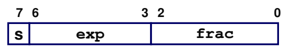


Tiny Floating-Point Example

8-bit floating-point representation

- The sign bit is in the most significant bit.
- The next four bits are the exponent, with a bias of 7.
- The last three bits are the `frac`

- Same general form as IEEE format
  - Normalized, denormalized
  - Representation of 0, NaN, infinity



Values Related to the Exponent

Exp	exp	E	2^E	
0	0000	-6	1/64	(denorms)
1	0001	-6	1/64	
2	0010	-5	1/32	
3	0011	-4	1/16	
4	0100	-3	1/8	
5	0101	-2	1/4	
6	0110	-1	1/2	
7	0111	0	1	
8	1000	+1	2	
9	1001	+2	4	
10	1010	+3	8	
11	1011	+4	16	
12	1100	+5	32	
13	1101	+6	64	
14	1110	+7	128	
15	1111	n/a		(inf, NaN)

Dynamic Range

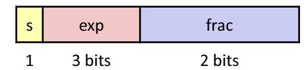
$v = (-1)^s \, M \, 2^E$
n: E = Exp – Bias
d: E = 1 – Bias

	s exp frac	E	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	← closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	← largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	← smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	← closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	← closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	← largest norm
	0 1111 000	n/a	inf	

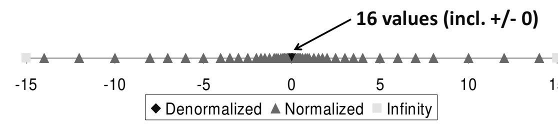
Distribution of Values

6-bit IEEE-like format

- **e = 3 exponent bits**
- **f = 2 fraction bits**
- **Bias is $2^{3-1}-1 = 3$**



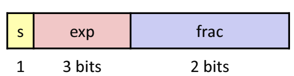
Notice how the distribution gets denser toward zero.



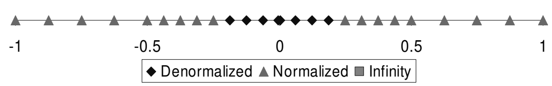
Distribution of Values (close-up view)

6-bit IEEE-like format

- **e = 3 exponent bits**
- **f = 2 fraction bits**
- **Bias is $2^{3-1}-1 = 3$**



Notice how the distribution gets denser toward zero (not all values shown).



Interesting Numbers

Description	exp	frac	Numeric Value
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
■ Single (float) ≈ 1.4×10^{-45}			
■ Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
■ Single (float) ≈ 1.18×10^{-38}			
■ Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
■ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
■ Single (float) ≈ 3.4×10^{38}			
■ Double ≈ 1.8×10^{308}			

Special Properties of Encoding

FP zero same as integer zero

- All bits = 0

Can (almost) use unsigned integer comparison

- Must first compare sign bits
- Must consider -0 = 0
- NaNs problematic
  - Will be greater than any other values
  - What should comparison yield?
- Otherwise OK
  - Denormalized vs. normalized
  - Normalized vs. infinity

Floating Point Operations: Basic Idea

$$x +_f y = \text{Round}(x + y)$$

$$x \times_f y = \text{Round}(x \times y)$$

Basic idea

- First compute exact result
- Make it fit into desired precision
  - Possibly overflow if exponent too large
  - Possibly round to fit into `frac`

Rounding

Rounding Modes (illustrated with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down (–∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

Closer Look at Round-To-Even

Default rounding mode

- Hard to get any other kind without dropping into assembly
- All others are statistically biased
  - Sum of set of positive numbers will consistently be over- or under-estimated
  - Need randomness

Applying to other decimal places / bit positions

- When exactly halfway between two possible values:
  - Round so that least significant digit is even
- E.g., round to nearest hundredth

1.2349999	1.23	(Less than halfway)
1.2350001	1.24	(Greater than halfway)
1.2350000	1.24	(Halfway—round up)
1.2450000	1.24	(Halfway—round down)

Rounding Binary Numbers

Binary fractional numbers

- "Even" when least significant bit is `0`
- Halfway when bits to right of rounding position = `100…`$_2$

Examples

- Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

Operands

$(-1)^{s1}\ M1\ 2^{E1}\ \ *\ \ (-1)^{s2}\ M2\ 2^{E2}$

Exact Result

$(-1)^{s}\ M\ 2^{E}$

- Sign *s*: $s1$ ^ $s2$
- Significand *M*: $M1 * M2$
- Exponent *E*: $E1 + E2$

Fixing

- If $M \geq 2$, shift *M* right, increment *E*
- If *E* out of range, overflow
- Round *M* to fit `frac` precision

Implementation

- Biggest chore is multiplying significands

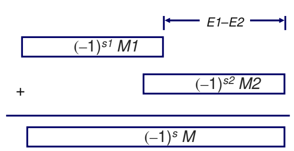
FP Addition

Operands

$(-1)^{s1}\ M1\ 2^{E1}$

$(-1)^{s2}\ M2\ 2^{E2}$

- Assume $E1 > E2$



Exact Result

$(-1)^{s}\ M\ 2^{E}$

- Sign *s*, significand *M*:
  - Result of signed align & add
- Exponent *E*: $E1$

Fixing

- If $M \geq 2$, shift *M* right, increment *E*
- if $M < 1$, shift *M* left *k* positions, decrement *E* by *k*
- If *E* out of range, overflow, denormalize, or generate 0
- Round *M* to fit `frac` precision

Mathematical Properties of FP Add

Compare to those of Abelian Group

- Closed under addition? *Yes*
  - But may generate infinity or NaN
- Commutative? *Yes*
- Associative? *No*
  - Overflow and inexactness of rounding
  - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
- 0 is additive identity? *Yes*
- Every element has additive inverse? *Almost*
  - Yes, except for infinities & NaNs

Monotonicity

- a ≥ b ⇒ a+c ≥ b+c? *Almost*
  - Except for infinities & NaNs

Mathematical Properties of FP Mult

Compare to Commutative Ring

- Closed under multiplication? *Yes*
  - But may generate infinity or NaN
- Multiplication Commutative? *Yes*
- Multiplication is Associative? *No*
  - Possibility of overflow, inexactness of rounding
  - Ex: `(1e20*1e20)*1e-20= inf, 1e20*(1e20*1e-20)= 1e20`
- 1 is multiplicative identity? *Yes*
- Multiplication distributes over addition? *No*
  - Possibility of overflow, inexactness of rounding
  - `1e20*(1e20-1e20)= 0.0, 1e20*1e20 – 1e20*1e20 = NaN`

Monotonicity

- $a \ge b$ & $c \ge 0 \Rightarrow a * c \ge b * c$? *Almost*
  - Except for infinities & NaNs

Floating Point in C

C Guarantees Two Levels

`float` single precision

`double` double precision

Conversions

- Casting between `int`, `float`, and `double` changes numeric values
- `Double` or `float` to `int`
  - Truncates fractional part
  - Like rounding toward zero
  - Not defined when out of range
    - » Generally saturates to TMin or TMax
- `int` to `double`
  - Exact conversion, as long as int has ≤ 53-bit word size
- `int` to `float`
  - Will round according to rounding mode

Answers to Floating-Point Puzzles

```
int x = …;
float f = …;
double d = …;
```

Assume neither `d` nor `f` is NAN

• `x == (int)(float) x`	No: 24-bit significand
• `x == (int)(double) x`	Yes: 53-bit significand
• `f == (float)(double) f`	Yes: increases precision
• `d == (float) d`	No: loses precision
• `f == -(-f)`	Yes: Just change sign bit
• `2/3 == 2/3.0`	No: 2/3 == 0
• `d < 0.0 ⇒ ((d*2) < 0.0)`	Yes, even for ∞!
• `d > f ⇒ -f > -d`	Yes!
• `d * d >= 0.0`	Yes, even for ∞!
• `(f+d)-d == f`	No: Not associative

Ariane 5

- Exploded 37 seconds after liftoff
- Cargo worth $500 million

Why

- Computed horizontal velocity as floating-point number
- Converted to 16-bit integer
- Worked OK for Ariane 4
- Overflowed for Ariane 5
  - Used same software

Summary

IEEE floating point has clear mathematical properties

- **Represents numbers of form $M \times 2^E$**
- **Can reason about operations independent of implementation**
  - **As if computed with perfect precision and then rounded**
- **Not the same as real arithmetic**
  - **Violates associativity/distributivity**
  - **Makes life difficult for compilers & serious numerical applications programmers**