Physics 31600 R. Wald
Classical Mechanics Autumn, 2002

Problem Set I

Due in class on Tuesday, October 15

1) (a) Consider a system with one degree of freedom and suppose its Lagrangian is a function of $\ddot{q}$ as well as q and $\dot{q}$, i.e. $L = L(q, \dot{q}, \ddot{q})$. Derive the Euler-Lagrange equations for this case, obtained by requiring $S[\gamma]$ to be an extremum with respect to variations which keep both q and $\dot{q}$ fixed at the endpoints. What is the maximum number of time derivatives of q that can appear in the equations of motion?

   (b) Obtain a Hamiltonian formulation of the equations of motion for this system as follows: Write $Q_1 = q, Q_2 = \dot{q}$ and define

$$P_2 = \partial L/\partial \ddot{q} = \partial L/\partial \dot{Q}_2 \qquad (*)$$

   Define the function H by,

$$H(Q_1, Q_2, P_1, P_2) = P_1 Q_2 + P_2 \ddot{Q}_2 - L(Q_1, Q_2, \ddot{Q}_2)$$

   where it is understood that $\ddot{Q}_2$ has been expressed as a function of (Q_1, Q_2, P_2) by solving $(*)$. Show that Hamilton's equations of motion for H are equivalent to the Euler-Lagrange equations derived in part (a).

2) Let $L(q, \dot{q}, t)$ be the Lagrangian of a particle moving in one dimension. Let $f(q, t)$ be an arbitrary function and define a new Lagrangian L' by adding the "total time derivative" of f to L, i.e.,

$$\begin{aligned} L'(q, \dot{q}, t) &= L(q, \dot{q}, t) + \frac{df}{dt} \\ &= L(q, \dot{q}, t) + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} \end{aligned}$$

   (a) Show that the equations of motion for L' are identical to those for L.

   (b) Relate the new canonical momentum, p', for L' to the old canonical momentum, p, for L. Express the new Hamiltonian $H'(q, p', t)$ for L' in terms of the old Hamiltonian $H(q, p, t)$ and f. Use the chain rule to express partial derivatives of H' with respect to (q, p') in terms of partial derivatives of H with respect to (q, p). Explicitly show, thereby, that the new Hamilton's equations for H' are equivalent to the old Hamilton's equations for H.

3) (a) A particle in ordinary 3-dimensional space, $\mathbf{R}^3$ is constrained to move on a 2- dimensional surface, S. Let (q_1, q_2) be coordinates on S. Show that the kinetic energy of the particle can be written in the form

$$T = \frac{1}{2} m \sum_{i,j} g_{ij}(q_1, q_2) \frac{dq_i}{dt} \frac{dq_j}{dt}$$

and express g_{ij} explicitly in terms of the vector function $\vec{x}(q_1, q_2)$ on S. (The quantities g_{ij} are the components of the *induced metric tensor* on S).

(b) For a system with n degrees of freedom having a Lagrangian of the form

$$L = \frac{1}{2} \sum_{i,j} g_{ij}(q_1, \ldots, q_n) \frac{dq_i}{dt} \frac{dq_j}{dt}$$

write down the Euler-Lagrange equations of motion.

(c) Show that the curves, γ, which satisfy the Euler-Lagrange equations of part (b) also extremize the distance along γ, $D[\gamma]$, between two points, where $D[\gamma]$ is given by

$$D[\gamma] = \int_{S_0}^{S_1} \left[\sum_{i,j} g_{ij}(q_1, \ldots, q_n) \frac{dq_i}{ds} \frac{dq_j}{ds} \right]^{\frac{1}{2}} ds$$

Such curves are called geodesics, and the combined results of parts (a), (b), and (c) show that a free particle confined to a surface, S moves on a geodesic in that surface.