Problem Set I
Due in class on Tuesday, October 15

1) (a) Consider a system with one degree of freedom and suppose its Lagrangian is a function of \( \ddot{q} \) as well as \( q \) and \( \dot{q} \), i.e. \( L = L(q, \dot{q}, \ddot{q}) \). Derive the Euler-Lagrange equations for this case, obtained by requiring \( S[\gamma] \) to be an extremum with respect to variations which keep both \( q \) and \( \dot{q} \) fixed at the endpoints. What is the maximum number of time derivatives of \( q \) that can appear in the equations of motion?

(b) Obtain a Hamiltonian formulation of the equations of motion for this system as follows: Write \( Q_1 = q, Q_2 = \dot{q} \) and define

\[
P_2 = \partial L/\partial \ddot{q} = \partial L/\partial \dot{Q}_2 \tag{\star}
\]

Define the function \( H \) by,

\[
H(Q_1, Q_2, P_1, P_2) = P_1 Q_2 + P_2 \ddot{Q}_2 - L(Q_1, Q_2, \ddot{Q}_2)
\]

where it is understood that \( \ddot{Q}_2 \) has been expressed as a function of \((Q_1, Q_2, P_2)\) by solving \((\star)\). Show that Hamilton’s equations of motion for \( H \) are equivalent to the Euler-Lagrange equations derived in part (a).

2) Let \( L(q, \dot{q}, t) \) be the Lagrangian of a particle moving in one dimension. Let \( f(q, t) \) be an arbitrary function and define a new Lagrangian \( L' \) by adding the “total time derivative” of \( f \) to \( L \), i.e.,

\[
L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{df}{dt}
\]

(a) Show that the equations of motion for \( L' \) are identical to those for \( L \).

(b) Relate the new canonical momentum, \( p' \), for \( L' \) to the old canonical momentum, \( p \), for \( L \). Express the new Hamiltonian \( H'(q, p', t) \) for \( L' \) in terms of the old Hamiltonian \( H(q, p, t) \) and \( f \). Use the chain rule to express partial derivatives of \( H' \) with respect to \((q, p')\) in terms of partial derivatives of \( H \) with respect to \((q, p)\). Explicitly show, thereby, that the new Hamilton’s equations for \( H' \) are equivalent to the old Hamilton’s equations for \( H \).
3) (a) A particle in ordinary 3-dimensional space, $\mathbb{R}^3$, is constrained to move on a 2-dimensional surface, $S$. Let $(q_1, q_2)$ be coordinates on $S$. Show that the kinetic energy of the particle can be written in the form

$$T = \frac{1}{2} m \sum_{i,j} g_{ij}(q_1, q_2) \frac{dq_i}{dt} \frac{dq_j}{dt}$$

and express $g_{ij}$ explicitly in terms of the vector function $\vec{x}(q_1, q_2)$ on $S$. (The quantities $g_{ij}$ are the components of the induced metric tensor on $S$).

(b) For a system with $n$ degrees of freedom having a Lagrangian of the form

$$L = \frac{1}{2} \sum_{i,j} g_{ij}(q_1, \ldots, q_n) \frac{dq_i}{dt} \frac{dq_j}{dt}$$

write down the Euler-Lagrange equations of motion.

(c) Show that the curves, $\gamma$, which satisfy the Euler-Lagrange equations of part (b) also extremize the distance along $\gamma$, $D[\gamma]$, between two points, where $D[\gamma]$ is given by

$$D[\gamma] = \int_{S_0}^{S_1} \left[ \sum_{i,j} g_{ij}(q_1, \ldots, q_n) \frac{dq_i}{ds} \frac{dq_j}{ds} \right]^{\frac{1}{2}} ds$$

Such curves are called geodesics, and the combined results of parts (a), (b), and (c) show that a free particle confined to a surface, $S$, moves on a geodesic in that surface.