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Problem Set I Solutions v1.1

1) (a) Consider a system with one degree of freedom and suppose its Lagrangian is a function of $\ddot{q}$ as well as q and $\dot{q}$, i.e. $L = L(q, \dot{q}, \ddot{q})$. Derive the Euler-Lagrange equations for this case, obtained by requiring $S[\gamma]$ to be an extremum with respect to variations which keep both q and $\dot{q}$ fixed at the endpoints. What is the maximum number of time derivatives of q that can appear in the equations of motion?

Solution: We straight forwardly compute the derivative of $S[\gamma]$.

$$S[\gamma + h] - S[\gamma] = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q} h + \frac{\partial L}{\partial \dot{q}} \dot{h} + \frac{\partial L}{\partial \ddot{q}} \ddot{h} \right] dt + O(h^2)$$

$$\rightarrow \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} \right] h dt + \frac{\partial L}{\partial \dot{q}} q \bigg|_{t_0}^{t_1} + \frac{\partial L}{\partial \ddot{q}} \dot{q} \bigg|_{t_0}^{t_1} + \left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \right) q \bigg|_{t_0}^{t_1}.$$

The last three terms vanish by the assumption that q and $\dot{q}$ are held fixed at the endpoints. Thus, the condition for the an extremum is that the bracketed term in the integral vanishes, yielding

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0.$$

The highest order time derivatives come from the last term, where we diff-eneritate L twice. Since in principle L could be higher than linear order in $\ddot{q}$ and it's being differentiated twice, we could get the fourth time derivative of q in the equations of motion. ■

(b) Obtain a Hamiltonian formulation of the equations of motion for this system as follows: Write $Q_1 = q, Q_2 = \dot{q}$ and define

$$P_2 = \partial L / \partial \ddot{q} = \partial L / \partial \dot{Q}_2 \qquad (*)$$

Define the function H by,

$$H(Q_1, Q_2, P_1, P_2) = P_1 Q_2 + P_2 \dot{Q}_2 - L(Q_1, Q_2, \dot{Q}_2)$$

where it is understood that $\dot{Q}_2$ has been expressed as a function of (Q_1, Q_2, P_2) by solving $(*)$. Show that Hamilton's equations of motion for H are equivalent to the Euler-Lagrange equations derived in part (a).

Solution: First off, we note that there was a typo in the problem sets given out. All references to $\ddot{Q}_2$ should have $\dot{Q}_2$. We apologize for not catching this sooner.

Next, before delving into the mechanics of computation, we pause to stress a couple of confusing issues. First is that P_1 **is not** defined as $\frac{\partial L}{\partial \dot{q}}$ or $\frac{\partial L}{\partial Q_1}$. In what's to come we won't actually need an explicit formula for P_1, and looking for one will actually make things more confusing. We will think of P_1 as simply a fourth indepedent variable in the set $\{Q_1, Q_2, P_1, P_2\}$. The second is that $\dot{Q}_2$ **is not** $\frac{d}{dt}Q_2$. It is simpliy some arbitrary function of Q_1, Q_2, and P_2 defined by $(*)$, above. Once we apply Hamilton's equations, it will turn out that $\dot{Q}_2 = \frac{d}{dt}Q_2$ for paths which satisfy the equations of motion. Note also that the function $\dot{Q}_2$ is independent of P_1 by construction. Now, for some Hamilton's equations:

$$\frac{\partial H}{\partial P_1} = \frac{d}{dt}Q_1 \Rightarrow Q_2 = \frac{d}{dt}Q_1$$

This follows since L is independent of P_1, and is non-vacuous in that it tells us that the independent variables Q_1 and Q_2 are related by a time derivative for paths which obey the equations of motion. Next:

$$\frac{\partial H}{\partial P_2} = \frac{d}{dt}Q_2 \Rightarrow \dot{Q}_2 + P_2 \left(\frac{\partial \dot{Q}_2}{\partial P_2}\right)_{Q_1,Q_2} - \left(\frac{\partial L}{\partial \dot{Q}_2}\right)_{Q_1,Q_2} \left(\frac{\partial \dot{Q}_2}{\partial P_2}\right)_{Q_1,Q_2} = \dot{Q}_2 = \frac{d}{dt}Q_2,$$

where the penultimate equality follows from the definition of P_2. As promised, the dynamical relationship between $\dot{Q}_2$ and $\frac{d}{dt}Q_2$ has come. Next up:

$$\frac{\partial H}{\partial Q_1} = -\frac{d}{dt}P_1 \Rightarrow$$

$$P_2 \left(\frac{\partial \dot{Q}_2}{\partial Q_1}\right)_{Q_2,P_2} - \left(\frac{\partial L}{\partial Q_1}\right)_{Q_2,\dot{Q}_2} - \left(\frac{\partial L}{\partial \dot{Q}_2}\right)_{Q_1,Q_2} \left(\frac{\partial \dot{Q}_2}{\partial Q_1}\right)_{Q_2,P_2} = -\left(\frac{\partial L}{\partial Q_1}\right)_{Q_2,\dot{Q}_2} = -\frac{d}{dt}P_1$$

and finally:

$$\frac{\partial H}{\partial Q_2} = -\frac{d}{dt}P_2 \Rightarrow$$

$$P_1 + P_2 \left(\frac{\partial \dot{Q}_2}{\partial Q_2}\right)_{Q_1,P_2} - \left(\frac{\partial L}{\partial Q_2}\right)_{Q_1,\dot{Q}_2} - \left(\frac{\partial L}{\partial \dot{Q}_2}\right)_{Q_1,Q_2} \left(\frac{\partial \dot{Q}_2}{\partial Q_2}\right)_{Q_1,P_2} =$$

$$= P_1 - \left(\frac{\partial L}{\partial Q_2}\right)_{Q_1,\dot{Q}_2} = -\frac{d}{dt}P_2,$$

where we've used the definition of P_2 to simplify both equations. Differentiating the last equation and plugging in the previous equation and the definition of P_2 yields:

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \dot{Q}_2}\right)_{Q_1,Q_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial Q_2}\right)_{Q_1,\dot{Q}_2} + \left(\frac{\partial L}{\partial Q_1}\right)_{Q_2,\dot{Q}_2} = 0.$$

Now we can finally use some dynamical equations. We have $Q_1 = q$, $Q_2 = \frac{d}{dt}Q_1 = \frac{d}{dt}q$ (from the first Hamilton's equation), and $\dot{Q}_2 = \frac{d}{dt}Q_2 = \frac{d^2}{dt^2}q$ (from the second Hamilton's equation). Now let us examine the first term when we make these substitutions:

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \dot{Q}_2}\right)_{Q_1,Q_2} = \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)_{q,\dot{q}} = \frac{d^2}{dt^2}\frac{\partial L(q,\dot{q},\ddot{q})}{\partial \ddot{q}}.$$

Note that the last equality would not have been true had it not been the case that Q_1 and Q_2, and hence q and $\dot{q}$, were being held fixed in the partial derivative. In the other two terms, the correct variables are also held fixed in the partial derivative so we can substitute with impunity to get

$$\frac{d}{dt^2}\frac{\partial L}{\partial \ddot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = 0,$$

which are the Lagrange's equations obtained in part (a). **Q.E.D.**

2) Let $L(q,\dot{q},t)$ be the Lagrangian of a particle moving in one dimension. Let $f(q,t)$ be an arbitrary function and define a new Lagrangian L' by adding the "total time derivative" of f to L, i.e.,

$$\begin{aligned} L'(q,\dot{q},t) &= L(q,\dot{q},t) + \frac{df}{dt} \\ &= L(q,\dot{q},t) + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} \end{aligned}$$

(a) Show that the equations of motion for L' are identical to those for L.

Solution: This is a straightfoward calculation:

$$\begin{aligned} \frac{\partial L'}{\partial q} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} &= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial\left(\frac{\partial f}{\partial q}\dot{q}\right)}{\partial q} - \frac{d}{dt}\frac{\partial\left(\frac{\partial f}{\partial q}\dot{q}\right)}{\partial \dot{q}} + \frac{\partial\frac{\partial f}{\partial t}}{\partial q} - \frac{d}{dt}\frac{\partial\frac{\partial f}{\partial t}}{\partial \dot{q}} \\ &= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial^2 f}{\partial q^2}\dot{q} - \frac{d}{dt}\frac{\partial f}{\partial q} + \frac{\partial^2 f}{\partial q\partial t} - \frac{d}{dt}0 \\ &= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial^2 f}{\partial q}\dot{q} - \left(\frac{\partial^2 f}{\partial^2 q}\dot{q} + \frac{\partial^2 f}{\partial t\partial q}\right) + \frac{\partial^2 f}{\partial q\partial t} \\ &= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \end{aligned}$$

using equality of mixed partials for C^2 functions. ■

(b) Relate the new canonical momentum, p', for L' to the old canonical momentum, p, for L. Express the new Hamiltonian $H'(q, p', t)$ for L' in terms of the old Hamiltonian $H(q, p, t)$ and f. Use the chain rule to express partial derivatives of H' with respect to (q, p') in terms of partial derivatives of H with respect to (q, p). Explicitly show, thereby, that the new Hamilton's equations for H' are equivalent to the old Hamilton's equations for H.

Solution: This is mostly calculation, but since we're taking partial derivatives in different coordinate systems, we need to be a bit careful. Calculating p' and H' is straightforward:

$$p' = \frac{\partial L'}{\partial \dot{q}} = p + \frac{\partial f}{\partial q}$$

$$H' = p'\dot{q} - L' = p\dot{q} - L + \frac{\partial f}{\partial q}\dot{q} - \left(\frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}\right) = H - \frac{\partial f}{\partial t}.$$

H' is naturally expressed in terms for new coordinates q' and p' related to the old coordinates via

$$p' = p + \frac{\partial f}{\partial q} \;\; ; \;\; q' = q \quad \Leftrightarrow \quad p = p' - \frac{\partial f}{\partial q}(q', t) \;\; ; \;\; q = q'.$$

Having solved for the old coordinates in terms of the new coordinates, we can now express the partial derivatives H' with respect to the new coordinates in terms of the old coordinates:

$$\frac{\partial H}{\partial p'} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial p'} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial p'} - \frac{\partial^2 f}{\partial q \partial t}\frac{\partial q}{\partial p'} - \frac{\partial^2 f}{\partial t^2}\frac{\partial t}{\partial p'} = \frac{\partial H}{\partial p}$$

$$\frac{\partial H'}{\partial q'} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial q'} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial q'} - \frac{\partial^2 f}{\partial q \partial t}\frac{\partial q}{\partial q'} - \frac{\partial^2 f}{\partial t^2}\frac{\partial t}{\partial q'} = \frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial^2 f}{\partial q' \partial q} - \frac{\partial^2 f}{\partial q \partial t}$$

We're not quite done with the second equation because there is a q' in the middle term on the right hand most side. This is a partial derivative holding p' and t constant. However, since the function is independent of p' and p, and t isn't being mucked around with, we can just as well regarding this as a partial derivative at fixed p and t, and thereby replace the q' with q. So we get

$$\frac{\partial H}{\partial q'} = \frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial^2 f}{\partial q^2} - \frac{\partial^2 f}{\partial q \partial t}.$$

Finally, we want to show that the Hamilton's equations for H' gives are equivalent to the Hamilton's equations for H. We therefore assume the former and show that theu imply the latter (in a way that also shows that the latter imply the former). First the easy one:

$$\frac{\partial H}{\partial p'} = \dot{q}' \Leftrightarrow \frac{\partial H}{\partial p} = q,$$

where we have used the expression for $\frac{\partial H}{\partial p'}$ computed above in substituting in the left hand side and the fact that $q' = q$ in substituting the right hand side. The second equation is only slightly more complicated:

$$\frac{\partial H}{\partial q'} = -\dot{p}' \Leftrightarrow \frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial^2 f}{\partial q^2} - \frac{\partial^2 f}{\partial q \partial t} = -\frac{d}{dt}\left(p + \frac{\partial f}{\partial q}\right)$$

using the partial calculated above and the definition of p'. Using the first Hamilon's equation and calculating $\dot{p}'$ explictly yields

$$\Leftrightarrow \frac{\partial H}{\partial q} - \frac{\partial H}{\partial p}\frac{\partial^2 f}{\partial q^2} - \frac{\partial^2 f}{\partial q \partial t} = -\left(\dot{p} + \frac{\partial f^2}{\partial q^2}\dot{q} + \frac{\partial f^2}{\partial t \partial q}\right) \Leftrightarrow \frac{\partial H}{\partial \dot{q}} = -\dot{p}.$$

Hence the equations of motion for the two Hamiltonia are equivalent. **Q.E.D.**

3) (a) A particle in ordinary 3-dimensional space, $\mathbf{R}^3$ is constrained to move on a 2- dimensional surface, S. Let (q_1, q_2) be coordinates on S. Show that the kinetic energy of the particle can be written in the form

$$T = \frac{1}{2}m\sum_{i,j} g_{ij}(q_1, q_2)\frac{dq_i}{dt}\frac{dq_j}{dt}$$

and express g_{ij} explicitly in terms of the vector function $\vec{x}(q_1, q_2)$ on S. (The quantities g_{ij} are the components of the *induced metric tensor* on S).

Solution: The kinetic energy of the particle is $T = \frac{1}{2}\sum_{I,J}^{3} \delta_{IJ}\frac{dx_I}{dt}\frac{dx_J}{dt}$. Using the fact fact that $x_I(t) = x_I(q_i(t))$ and the chain rule, we find that

$$T = \frac{1}{2}\sum_{I,J=1}^{3} \delta_{IJ}\left(\sum_{i=1}^{2}\frac{\partial x_I}{\partial q_i}\frac{dq_i}{dt}\right)\left(\sum_{j=1}^{2}\frac{\partial x_J}{\partial q_j}\frac{dq_j}{dt}\right) = \sum_{i,j=1}^{2} g_{ij}\frac{dq_i}{dt}\frac{dq_j}{dt},$$

where

$$g_{ij} = \sum_{I=1}^{3}\left(\frac{\partial X_I}{\partial q_i}\right)\left(\frac{\partial X_I}{\partial q_j}\right). \quad \blacksquare$$

(b) For a system with n degrees of freedom having a Lagrangian of the form

$$L = \frac{1}{2}\sum_{i,j} g_{ij}(q_1, \ldots, q_n)\frac{dq_i}{dt}\frac{dq_j}{dt}$$

write down the Euler-Lagrange equations of motion.

Solution: Clearly,

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j=1}^{2}\frac{\partial g_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j.$$

Now,

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2}\sum_{i,j=1}^{2} g_{ij}\left(\delta_{ik}\dot{q}_j + \delta_{jk}\dot{q}_i\right) = \sum_{i=1}^{2} g_{ik}\dot{q}_i,$$

where we have used the symmetry of g_{ij} in the last step. So, the Euler-Lagrange equations are

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \frac{1}{2}\sum_{i,j=1}^{2}\frac{\partial g_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \sum_{i,j=1}^{2}\frac{\partial g_{ik}}{\partial q_j}\dot{q}_j\dot{q}_i - \sum_{i=1}^{2} g_{ik}\ddot{q}_i = 0. \quad \blacksquare$$

(c) Show that the curves, γ, which satisfy the Euler-Lagrange equations of part (b) also extremize the distance along $\gamma, D[\gamma]$, between two points, where $D[\gamma]$ is given by

$$D[\gamma] = \int_{S_0}^{S_1}\left[\sum_{i,j} g_{ij}(q_1,\ldots,q_n)\frac{dq_i}{ds}\frac{dq_j}{ds}\right]^{\frac{1}{2}} ds$$

Such curves are called geodesics, and the combined results of parts (a), (b), and (c) show that a free particle confined to a surface, S moves on a geodesic in that surface.

Solution: Geodesics extremize the action of the Lagrangian $\tilde{L} = \sqrt{2L}$. Let us begin by taking a look at its Euler-Lagrange equations:

$$\frac{\partial \tilde{L}}{\partial q} - \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}} = \frac{1}{\tilde{L}}\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{1}{\tilde{L}}\frac{\partial L}{\partial \dot{q}} = \frac{1}{\tilde{L}}\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right] + \frac{1}{\tilde{L}^3}\frac{dL}{dt} = 0$$

Let γ be an extremum of L. Since it satisfies the Euler-Lagrange equation of a free particle, its kinetic energy ($= L$) is conserved. Thus, there are two cases: either $L(\gamma(t)) = 0$ for all time or $L(\gamma(t))$ is non-zero. If $L(\gamma(t))$ is zero, then clearly γ is an extremum of $\tilde{L}$ since $D[\gamma] = 0$ and D is positive definite. If $L(\gamma(t)) \neq 0$, then the equations above are meaningful and need to be examined. The first terms vanishes since γ satisifies the Euler-Lagrange equations for L. The second term vanishes because L is conserved. Therefore γ is again a geodesic. Note that the converse of this statement is not true: there are extrema of $\tilde{L}$ which are not extrema of L. This is because $\tilde{L}$ is paramterization indepedent, whereas L is not. In the language of differential geometry, L gives the *affinely-parameterized* geodesic equations, whereas $\tilde{L}$ gives the true geodesic equation. $\blacksquare$