Physics 31600 | R. Wald
Classical Mechanics | Autumn, 2002

Problem Set II

Due in class on Tuesday, October 22

1) Let $L(q, \dot{q}; t)$ be a Lagrangian [where, as in class, "q" stands for $(q_1, \ldots, q_n)$]. Suppose we introduce new coordinates $(Q_1(q), \ldots, Q_n(q))$ on configuration space. Relate the new momenta, P, to the "old" momenta p and show that $\sum_i P_i \dot{Q}_i = \sum_i p_i \dot{q}_i$. (For the purposes of this problem, it is convenient to view all quantities as functions of the independent variables $(q, \dot{q})$.)

2) (a) Show that the Euler-Lagrange equations for the Lagrangian

$$L = \frac{1}{2} m \left| \frac{d\vec{x}}{dt} \right|^2 - e\phi + \frac{e}{c} \vec{A} \cdot \frac{d\vec{x}}{dt}$$

yield the usual Lorentz force equations of motion of a charged particle in an electromagnetic field.

(b) Obtain the corresponding Hamiltonian formulation of the problem. Write out Hamilton's equation of motion and show explicitly that they also are equivalent to the usual Lorentz force law.

3) In the context of special relativity, it is much more in keeping with the "covariant" nature of the theory to treat all four spacetime coordinates (t, x, y, z) on an equal footing, and thus to describe particle motion as a path $t(\lambda), x(\lambda), y(\lambda), z(\lambda)$ in a 4-dimensional configuration space (with λ an arbitrary parameter along the path) rather than as a curve $x(t), y(t), z(t)$ in a 3-dimensional configuration space (with t the time coordinate of a particular global inertial coordinate system).

(a) Show that the Lagrangian

$$L = -m \left[\left(\frac{dt}{d\lambda} \right)^2 - \left| \frac{d\vec{x}}{d\lambda} \right|^2 \right]^{\frac{1}{2}}$$

yields the correct equations of motion for a free particle. (In keeping with the above remark, treat (t, x, y, z) as the "degrees of freedom" and λ as "time".)

(b) Show that the conjugate momenta satisfy the relation

$$p_t^2 - (p_x^2 + p_y^2 + p_z^2) = m^2$$

and, thus, are not independent, i.e. one cannot eliminate the $\dot{q}$'s in favor of p's.

(c) Nevertheless, obtain a (constrained) Hamiltonian formulation for the free relativistic particle by the procedure described in class, with $\alpha = dt/d\lambda$.