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Classical Mechanics Autumn, 2002

Problem Set II Solutions

1) Let $L(q, \dot{q}; t)$ be a Lagrangian [where, as in class, "q" stands for $(q_1, \ldots, q_n)$]. Suppose we introduce new coordinates $(Q_1(q), \ldots, Q_n(q))$ on configuration space. Relate the new momenta, P, to the "old" momenta p and show that $\sum_i P_i \dot{Q}_i = \sum_i p_i \dot{q}_i$. (For the purposes of this problem, it is convenient to view all quantities as functions of the independent variables $(q, \dot{q})$.)

The momentum has the form:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial \dot{Q}_j}{\partial \dot{q}_i} + \frac{\partial L}{\partial Q_j}\frac{\partial Q_j}{\partial \dot{q}_i} \tag{1}$$

(There is an implied sum over double indices in the above equation and those to follow.) But the second term is zero because the Q_i's are functions only of q_i. Because of this we also have that:

$$\dot{Q}_j = \frac{\partial Q_j}{\partial q_i}\dot{q}_i \tag{2}$$

So:

$$\frac{\partial \dot{Q}_j}{\partial \dot{q}_i} = \frac{\partial Q_j}{\partial q_i} \tag{3}$$

Thus the new momentum P_i is related to the old by the formula:

$$p_i = \frac{\partial L}{\partial \dot{Q}_j}\frac{\partial Q_j}{\partial q_i} = P_j \frac{\partial Q_j}{\partial q_i} \tag{4}$$

We also immediately have:

$$p_i \dot{q}_i = P_j \frac{\partial Q_j}{\partial q_i}\dot{q}_i = P_j \dot{Q}_j \tag{5}$$

Recall that there is an implied sum over double indices.

2) (a) Show that the Euler-Lagrange equations for the Lagrangian

$$L = \frac{1}{2}m\left|\frac{d\vec{x}}{dt}\right|^2 - e\phi + \frac{e}{c}\vec{A}\cdot\frac{d\vec{x}}{dt}$$

yield the usual Lorentz force equations of motion of a charged particle in an electromagnetic field.

(b) Obtain the corresponding Hamiltonian formulation of the problem. Write out Hamilton's equation of motion and show explicitly that they also are equivalent to the usual Lorentz force law.

(a) First calculate the conjugate momenta:

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c}A_i \tag{6}$$

The "force" term is:

$$\frac{\partial L}{\partial x_i} = -e\frac{\partial \phi}{\partial x_i} + \frac{e}{c}\dot{\vec{x}} \cdot \frac{\partial \vec{A}}{\partial x_i} \tag{7}$$

So the Euler-Langrange equations are:

$$\frac{d}{dt}(m\dot{x}_i + \frac{e}{c}A_i) = -e\frac{\partial \phi}{\partial x_i} + \frac{e}{c}\dot{\vec{x}} \cdot \frac{\partial \vec{A}}{\partial x_i} \tag{8}$$

Subtracting the time derivative of $\vec{A}$ from both sides and using the chain rule we have:

$$m\ddot{x}_i = [-e\frac{\partial \phi}{\partial x_i} - \frac{e}{c}\frac{\partial A_i}{\partial t}] + \frac{e}{c}\dot{\vec{x}} \cdot \frac{\partial \vec{A}}{\partial x_i} - \frac{e}{c}(\dot{\vec{x}} \cdot \nabla)A_i \tag{9}$$

The first term in brackets is eE_i, where $\vec{E}$ is the electric field. The second term can be calculated as follows:

$$\dot{x_j}\partial_i A_j - \dot{x_j}\partial_j A_i = \dot{x_j}\partial_l A_k(\delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki}) = \dot{x_j}\partial_l A_k(\epsilon_{lkm}\epsilon_{ijm}) \tag{10}$$

But $\partial_l A_k \epsilon_{lkm} = B_m$ and $\dot{x_j}B_m\epsilon_{ijm} = (\dot{\vec{x}} \times \vec{B})_i$, so:

$$m\ddot{\vec{x}} = e\vec{E} + \frac{e}{c}\dot{\vec{x}} \times \vec{B} \tag{11}$$

which is the Lorentz Force Law.

(b) From equation 6 above we know that:

$$\vec{p} = m\dot{\vec{x}} + \frac{e}{c}\vec{A} \tag{12}$$

Thus,

$$H = \vec{p} \cdot \dot{\vec{x}} - L = \vec{p} \cdot \frac{\vec{p} - \frac{e}{c}\vec{A}}{m} - [\frac{m}{2}(\frac{\vec{p} - \frac{e}{c}\vec{A}}{m})^2 - e\phi + \frac{e}{c}\vec{A} \cdot (\frac{\vec{p} - \frac{e}{c}\vec{A}}{m})] \tag{13}$$

which gives

$$\begin{aligned} H &= (1 - \frac{1}{2})\frac{p^2}{m} + (-1 + 1 - 1)\frac{e\vec{p} \cdot \vec{A}}{mc} + (-\frac{1}{2} + 1)\frac{(eA)^2}{mc^2} + e\phi \\ &= \frac{p^2}{2m} - \frac{e\vec{p} \cdot \vec{A}}{mc} + \frac{(eA)^2}{2mc^2} + e\phi \end{aligned} \tag{14}$$

Hamilton's equations are:

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} - \frac{eA_i}{mc} \tag{15}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = +\frac{e\vec{p}}{mc} \cdot \frac{\partial \vec{A}}{\partial x_i} - \frac{e^2}{mc^2}\vec{A} \cdot \frac{\partial \vec{A}}{\partial x_i} - e\frac{\partial \phi}{\partial x_i} \tag{16}$$

Using the equation for $\dot{x}$ we can write the $\dot{p}$ equation as:

$$\frac{d}{dt}(m\dot{x}_i + \frac{eA_i}{c}) = \dot{\vec{x}} \cdot \frac{e}{c}\frac{\partial \vec{A}}{\partial x_i} - e\frac{\partial \phi}{\partial x_i} \tag{17}$$

This is the same as equation 8 above; thus Hamilton's equations also reduce to the Lorentz Force Law.

3) In the context of special relativity, it is much more in keeping with the "covariant" nature of the theory to treat all four spacetime coordinates (t, x, y, z) on an equal footing, and thus to describe particle motion as a path $t(\lambda), x(\lambda), y(\lambda), z(\lambda)$ in a 4-dimensional configuration space (with λ an arbitrary parameter along the path) rather than as a curve $x(t), y(t), z(t)$ in a 3-dimensional configuration space (with t the time coordinate of a particular global inertial coordinate system).

(a) Show that the Lagrangian

$$L = -m\left[\left(\frac{dt}{d\lambda}\right)^2 - \left|\frac{d\vec{x}}{d\lambda}\right|^2\right]^{\frac{1}{2}}$$

yields the correct equations of motion for a free particle. (In keeping with the above remark, treat (t, x, y, z) as the "degrees of freedom" and λ as "time".)

(b) Show that the conjugate momenta satisfy the relation

$$p_t^2 - (p_x^2 + p_y^2 + p_z^2) = m^2$$

and, thus, are not independent, i.e. one cannot eliminate the $\dot{q}$'s in favor of p's.

(c) Nevertheless, obtain a (constrained) Hamiltonian formulation for the free relativistic particle by the procedure described in class, with $\alpha = dt/d\lambda$.

(a) First, let's define some simpler notation:

$$\dot{q}^\mu = u^\mu = (\frac{dt}{d\lambda}, \frac{dx}{d\lambda}, \frac{dy}{d\lambda}, \frac{dz}{d\lambda}) \tag{18}$$

$$\dot{q}_\mu = u_\mu = (\frac{dt}{d\lambda}, -\frac{dx}{d\lambda}, -\frac{dy}{d\lambda}, -\frac{dz}{d\lambda}) \tag{19}$$

where q^μ are the coordinates and u^μ are the first derivatives with respect to λ. Then the Lagrangian is now expressed as:

$$L = -m[u^\mu u_\mu]^{\frac{1}{2}} \tag{20}$$

The momenta are:

$$p_\mu = \frac{\partial L}{\partial u^\mu} = -m u_\mu [u^\nu u_\nu]^{-\frac{1}{2}} \tag{21}$$

Since the coordinates do not explicitly appear in L, the equation of motion is:

$$\frac{d}{d\lambda}(-m u_\mu [u^\nu u_\nu]^{-\frac{1}{2}}) = 0 \tag{22}$$

which says the term in parenthesis is a constant, which we will call P_μ.

$$P_\mu = -m u_\mu [u^\nu u_\nu]^{-\frac{1}{2}} \tag{23}$$

Note that this expression is independent of the parametrization λ along the world-line. If I choose a new $\lambda' = f(\lambda)$ then P_μ is:

$$P_\mu = -m\beta u_\mu [\beta u^\nu \beta u_\nu]^{-\frac{1}{2}} = -m u_\mu [u^\nu u_\nu]^{-\frac{1}{2}} \tag{24}$$

where $\beta = \frac{d\lambda'}{d\lambda}$. So we can choose λ equal to the proper time τ and equation 23 becomes:

$$P_\mu = -m \frac{dq_\mu}{d\tau} \tag{25}$$

which is the familiar expression for the (constant) momentum of a relativistic particle.

The parameter λ is affine if the function $f(\lambda) = 0$ in the following equation:

$$u^\mu \partial_\mu u^\nu = f(\lambda) u^\nu \tag{26}$$

Carrying out the derivative in equation 22, we will get (after dividing by $-m$):

$$\dot{u_\mu} [u^\nu u_\nu]^{-\frac{1}{2}} - u_\mu [u^\nu u_\nu]^{-\frac{3}{2}} u_\sigma \dot{u^\sigma} = 0 \tag{27}$$

This leads to:

$$\dot{u_\mu} = [\frac{u_\sigma \dot{u^\sigma}}{u^\nu u_\nu}] u_\mu \tag{28}$$

After noticing that the left-hand side of equation 24 is $\dot{u^\nu}$, we see that the function in brackets is $f(\lambda)$. Thus, choosing λ affine gives $\dot{u^\mu} = 0$, which allows us to set $u^\mu u_\mu$ to a constant (which is 1 if λ is the proper time). It is easy to show that if λ is not affine, one can choose a new λ' that is affine.

(b) Using equation 21, we see that:

$$p^\mu p_\mu = m^2 u_\mu u^\mu [u^\nu u_\nu]^{-\frac{1}{2}} [u^\sigma u_\sigma]^{-\frac{1}{2}} = m^2 u_\mu u^\mu [u^\nu u_\nu]^{-1} = m^2 \tag{29}$$

(c) Since the p_μ are not independent, we cannot eliminate all of the u_μ in favor of them. We choose one of them ($\alpha = \frac{dt}{d\lambda}$) to serve as a non-dynamical constraining variable. Thus we have the Hamiltonian:

$$H = p_0\alpha + p_i u^i - L(\alpha, p_i) \tag{30}$$

where i runs over the spatial variables. Define $\gamma = \alpha[u^\mu u_\mu]^{-\frac{1}{2}}$ (note that this has no α dependence.) Thus:

$$H = p_0\alpha - \frac{p_i p^i}{m}[u^\mu u_\mu]^{\frac{1}{2}} + m[u^\mu u_\mu]^{\frac{1}{2}} = p_0\alpha - \frac{p_i p^i}{m}\frac{\alpha}{\gamma} + m\frac{\alpha}{\gamma} \tag{31}$$

Note that $-p_i p^i = \vec{p}^2$ and that:

$$(\gamma m)^2 = \frac{m^2}{1-\vec{v}^2} = m^2\frac{1-\vec{v}^2+\vec{v}^2}{1-\vec{v}^2} = m^2 + m^2(\gamma\vec{v})^2 = m^2 + \vec{p}^2 \tag{32}$$

Thus:

$$H = p_0\alpha + \alpha\sqrt{\vec{p}^2 + m^2} \tag{33}$$

So Hamilton's equations read:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{\sqrt{\vec{p}^2+m^2}} = \frac{p_i}{\gamma m} \tag{34}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \tag{35}$$

$$\frac{\partial H}{\partial \alpha} = p_0 + \sqrt{\vec{p}^2+m^2} = 0 \tag{36}$$