Physics 31600 R. Wald
Classical Mechanics Autumn, 2002

# Problem Set III

Due in class on Tuesday, October 29

1) As seen in the previous problem set, the Lagrangian for a (non-relativistic) particle of mass, $m$, and charge, $e$, in a magnetic field $\vec{B}$ is

$$L = \frac{1}{2}m\left|\frac{d\vec{x}}{dt}\right|^2 + \frac{e}{c}\vec{A}\cdot\frac{d\vec{x}}{dt}$$

Consider the vector potential $\vec{A} = \frac{1}{2}\vec{B}_0 \times \vec{x}$ (with $\vec{B}_0$ constant), corresponding to a uniform magnetic field, $\vec{B}_0$.

(a) Identify all of the (one-parameter groups of) spatial symmetries of this Lagrangian.

(b) Choose coordinates adapted to these symmetries.

(c) Write down all the constants of motion associated with these symmetries as well as the constant of motion associated with the time translation symmetry.

(d) Obtain the general solution to the equations of motion. (It will simplify your analysis to make use of the freedom available in choosing the origin of coordinates relative to the initial position and velocity of the particle.)

2) A particle of mass, $m$, moving in ordinary, 3-dimensional space, is acted upon by a "central potential", i.e. the potential, $V$, depends only upon $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

(a) Write down the Lagrangian, $L$, for the problem in spherical polar coordinates $(r, \theta, \phi)$.

(b) Explicitly obtain expressions for the 3 constants of motion which arise from the invariance of $L$ under rotations about the $x, y$, and $z$ axes.

(c) Derive an equation expressing $\dot{r}$ as a function of $r$ and constants of the motion. (This equation, together with similar equations for $\phi$ and $\theta$ obtained from part (b), reduces the general central force problem "to quadratures".)

3) Let $W$ be a finite dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$, and let $U : W \to W$ be a linear map.

(a) Show that $U$ preserves the norm of all vectors if and only if $U^\dagger U = I$. For a real vector space, such a $U$ is called an *orthogonal* map; for a complex vector space, $U$ is called a *unitary* map.

(b) For an orthogonal or unitary map, $U$, show that (i) Any eigenvalue, $\lambda$, of $U$ satisfies $|\lambda| = 1$. (ii) Any two eigenvectors of $U$ with distinct eigenvalues must be orthogonal. (iii) If a subspace $S$ satisfies $U[S] \subset S$, then $U[S^\perp] \subset S^\perp$ (Hint: Show that $U[S] = S$ and hence $U^{-1}[S] = S$.)