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Problem Set III Solutions v1.2

1) As seen in the previous problem set, the Lagrangian for a (non-relativistic) particle of mass, m, and charge, e, in a magnetic field $\vec{B}$ is

$$L = \frac{1}{2}m\left|\frac{d\vec{x}}{dt}\right|^2 + \frac{e}{c}\vec{A}\cdot\frac{d\vec{x}}{dt}$$

Consider the vector potential $\vec{A} = \frac{1}{2}\vec{B}_0 \times \vec{x}$ (with $\vec{B}_0$ constant), corresponding to a uniform magnetic field, $\vec{B}_0$.

(a) Identify all of the (one-parameter groups of) spatial symmetries of this Lagrangian.

Solution: Translations are a symmetry of the kinetic energy, and translations along the $\vec{B}_0$ axis are symmetries of the potential energy (since $\vec{B}_0 \times \vec{B}_0 = 0$) so translations in the direction of $\vec{B}_0$ are one symmetry. Further, rotations are also a symmetry of the kinetic energy. For rotations about $\vec{B}_0$, and only for rotations about $\vec{B}_0$, we have $\vec{B}_0 \times (R\vec{x}) = R(\vec{B}_0 \times \vec{x})$, where R is the rotation matrix. Hence the potential term changes as $\vec{A}\cdot\vec{v} \to (R\vec{A})\cdot(R\vec{v}) = \vec{A}\cdot\vec{v}$ since rotations are orthogonal transformations. Thus rotations about $\vec{B}_0$ are also symmetries. These are the only continuous spatial symmetries. ■

(b) Choose coordinates adapted to these symmetries.

Solution: It is natural to choose cylindrical coordinates, with $\vec{B}_0 = B_0\hat{z}$. The Lagrangian should then be independent of z and θ (see part (c)). ■

(c) Write down all the constants of motion associated with these symmetries as well as the constant of motion associated with the time translation symmetry.

Solution: When the Lagrangian is expressed in terms of coordinates adapted to its symmetries, the conserved quantities will be the momenta conjugate to its cyclic coordinates. First, we expand the Lagrangian in terms of components:

$$L = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] + \frac{e}{2c}B_0(x\dot{y} - y\dot{x}).$$

Now, we switch to cylindrical coordinates using:

$$x = \rho\cos\theta \Rightarrow \dot{x} = \dot{\rho}\cos\theta - \rho\sin\theta\dot{\theta}$$

$$y = \rho\sin\theta \Rightarrow \dot{y} = \dot{\rho}\sin\theta + \rho\cos\theta\dot{\theta},$$

which gives

$$L = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right] + \frac{e}{2c}B_0\rho^2\dot{\theta}.$$

As promised, the Lagrangian is independent of z and θ. So the conserved quantitities are:

$$P_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} = \text{const}$$

and

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\rho^2\left[\dot{\theta} + \frac{eB_0}{2mc}\right] = \text{const}.$$

Finally, we want the constant of motion of associated with time translation invariance, also known as the Hamiltonian. First we'll need to momentum conjugate to ρ:

$$P_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}.$$

Thus, the Hamiltonian is:

$$H = \sum_i P_i\dot{q}_i - L = \frac{1}{2}m\left[\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\theta}^2\right] = T.$$

The Hamiltonian is equal to the kinetic energy because magnetic fields do no work; the "potential" term $\vec{A}\cdot\vec{v}$ does not represent potential energy. ■

(d) Obtain the general solution to the equations of motion. (It will simplify your analysis to make use of the freedom available in choosing the origin of coordinates relative to the initial position and velocity of the particle.)

Solution: Since we have 3 first integrals of the motion, it is helpful to see what they do before we start hacking away with the Euler-Lagrange equations. Indeed, we can plug the conservation of z-momentum into the Hamiltonian and solve for $\dot{\rho}$, yielding

$$\dot{\rho} = \pm\sqrt{\frac{2H}{m} - \frac{P_z^2}{m^2} - \rho^2\dot{\theta}^2}.$$

Now if we choose our coordinate system so that $\rho(0) = 0$, then $P_\theta(0) = 0$. But P_θ is a conserved quantity, and hence stays zero for all time, which means that either

$$\rho(t) = 0 \quad \text{or} \quad \dot{\theta}(t) = -\frac{eB_0}{2mc}.$$

In the first case, the particle just goes along the z-axis with constant velocity. In the second, we can plug $\dot{\theta}$ into the equation for $\dot{\rho}$ to obtain:

$$\dot{\rho} = \pm\sqrt{\frac{2H}{m} - \frac{P_z^2}{m^2} - \rho^2\frac{e^2B_0^2}{4m^2c^2}}.$$

Now we let

$$u = \rho\frac{eB_0}{2mc} \quad ; \quad a^2 = \frac{2H}{m} - \frac{P_z^2}{m^2} > 0$$

and separate the variables to get

$$\int_{u(0)}^{u(t)} \frac{2mc}{eB_0} \frac{\pm du}{\sqrt{a^2 - u^2}} = \pm \frac{2mc}{eB_0} \sin^{-1} \frac{u}{a}\Big|_{u(0)}^{u(t)} = t = \int_0^t dt'.$$

Since $\rho(0) = 0 \Rightarrow u(0) = 0$, we find that

$$\rho(t) = \pm \frac{2mc}{eB_0} \sqrt{\frac{2H}{m} - \frac{P_z^2}{m^2}} \sin\left(\frac{eB_0 t}{2mc}\right).$$

Note that the special case of $\rho(t) = 0$ mentioned above corresponds to (from the $\dot{\rho}$ equation) $2H = P_z^2/m$. This equality means that the amplitude of the sinusoidal solution vanishes, in complete agreement. This energy condition also corresponds with our intuition: when the initial velocity is entirely along the B-field, so all the energy is in the z direction, the particle experiences no force from the B-field. From the conservation equations we can also find z and θ:

$$z(t) = \frac{P_z}{m} t + z(0) \quad ; \quad \theta(t) = -\frac{eB_0}{2mc} t + \theta(0).$$

Since $\rho \to -\rho$ and $\theta \to \theta + \pi$ are the same transformation, we can consider just the $+$ solution for $\rho(t)$ and absorb the orientation into $\theta(0)$. Finally, we show that this solution corresponds to helical motion. Taking, for example, the $\theta(0) = 0$ solution, its x and y coordinates are

$$(x(t), y(t)) = \frac{2mc}{eB_0} \sqrt{\frac{2H}{m} - \frac{P_z^2}{m^2}} \left(\sin\left(\frac{eB_0 t}{2mc}\right) \cos\left(\frac{eB_0 t}{2mc}\right), -\sin\left(\frac{eB_0 t}{2mc}\right)^2 \right),$$

so its distance from the axis of the helix $\left(0, -\frac{1}{2}\frac{2mc}{eB_0}\sqrt{\frac{2H}{m} - \frac{P_z^2}{m^2}}\right)$ is

$$\frac{2mc}{eB_0} \sqrt{\frac{2H}{m} - \frac{P_z^2}{m^2}} \sqrt{\sin^2 \omega t \cos^2 \omega t + \left(\sin^2 \omega t - \frac{1}{2}\right)^2} = \frac{1}{2} \frac{2mc}{eB_0} \sqrt{\frac{2H}{m} - \frac{P_z^2}{m^2}}.$$

Hence, the particle moves in a (possibly degenerate) helix. ■

Note: Many students tried to solve this problem using the EL equations. That is certainly possible, but you *must* make use of the conservation equations to make the equations of motion tractable. For example, having made a choice of $\rho(0) = 0$ to make $\dot{\theta}$ constant, you would then find that the EL equation for ρ is $\ddot{\rho} = \text{const} \times \rho$, which gives the above sinusoidal solutions.

2) A particle of mass, m, moving in ordinary, 3-dimensional space, is acted upon by a "central potential", i.e. the potential, V, depends only upon $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

(a) Write down the Lagrangian, L, for the problem in spherical polar coordinates (r, θ, ϕ).

Solution: Our Lagrangian is

$$\frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]-V(r).$$

Spherical and Cartesian coordinates are related by

$$x=r\sin\theta\cos\varphi\Rightarrow\dot{x}=\dot{r}\sin\theta\cos\varphi+r\cos\theta\cos\varphi\dot{\theta}-r\sin\theta\sin\varphi\dot{\varphi}$$

$$y=r\sin\theta\sin\varphi\Rightarrow\dot{y}=\dot{r}\sin\theta\sin\varphi+r\cos\theta\sin\varphi\dot{\theta}+r\sin\theta\cos\varphi\dot{\varphi}$$

$$z=r\cos\theta\Rightarrow\dot{z}=\dot{r}\cos\theta-r\sin\theta\dot{\theta}.$$

Plugging these into the Lagrangian yields

$$L=\frac{1}{2}m\left[\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\varphi}^2\right]-V(r)\quad\blacksquare$$

(b) Explicitly obtain expressions for the 3 constants of motion which arise from the invariance of L under rotations about the x, y, and z axes.

Solution: The constants of the motion will be, of course, the angular momentum, but let us see how Noether's theorem gives them to us. Let us first consider rotations about the z-axis. These can be easily expressed as the 1-parameter family:

$$r_s=r\quad;\quad\theta_s=\theta\quad;\quad\varphi_s=\varphi+s.$$

Noether's theorem says that the conserved quantity is

$$L_z=\sum_i\frac{\partial L}{\partial\dot{q}_i}\frac{dq_i}{ds}\bigg|_{s=0}=\frac{\partial L}{\partial\dot{\varphi}}\;\frac{d(\varphi+s)}{ds}\bigg|_{s=0}=mr^2\sin^2\theta\dot{\varphi}.$$

Notice that the machinery of Noether's theorem wasn't really necessary in this case; we could have used the fact that the Lagrangian was cyclic in φ to obtain a conserved quantity. Now let us consider rotations about the x-axis, which are easily expressed in Cartesian coordinates by the 1-parameter family

$$x_s=x\quad;\quad y_s=y\cos s-z\sin s\quad;\quad z_s=z\cos s+y\sin s.$$

Ultimately, we want the conserved quantity expressed in spherical coordinates. Thus, we have two choices: we can either re-express the 1-parameter family in spherical coordinates and compute the conserved quantity directly, or we can first compute the quantity, then convert to polar coordinates. It turns out that the second option is easier. So

$$L_x=\frac{\partial L}{\partial\dot{y}}\;\frac{d(y\cos s-z\sin s)}{ds}\bigg|_{s=0}+\frac{\partial L}{\partial\dot{z}}\;\frac{d(z\cos s+y\sin s)}{ds}\bigg|_{s=0}=m\left[y\dot{z}-z\dot{y}\right],$$

which is indeed the angular momentum about x. Converting to polar coordinates yields:

$$L_x = -mr^2 \left[\sin\varphi\dot{\theta} + \sin\theta\cos\theta\cos\varphi\dot{\varphi} \right].$$

By the same procedure with the 1-parameter family

$$x_s = x\cos s + z\sin s \quad ; \quad y_s = y \quad ; \quad z_s = z\cos s - x\sin s$$

we obtain

$$L_y = m\left[z\dot{x} - x\dot{z}\right] = mr^2 \left[\cos\varphi\dot{\theta} - \sin\theta\cos\theta\sin\varphi\dot{\varphi} \right]. \quad \blacksquare$$

(c) Derive an equation expressing $\dot{r}$ as a function of r and constants of the motion. (This equation, together with similar equations for ϕ and θ obtained from part (b), reduces the general central force problem "to quadratures".)

Solution: Since L is time-independent and of the standard form $T - V$, the total energy E is conserved:

$$E = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right] + V(r) = \text{const.}$$

Next, note that

$$L^2 = L_x^2 + L_y^2 + L_z^2 = m^2r^4\left[\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right],$$

so we can write

$$E = \frac{1}{2}\left[\dot{r}^2 + \frac{L^2}{mr^2}\right] + V(r).$$

Inverting this equation yields:

$$\dot{r} = \pm\sqrt{\frac{2(E-V)}{m} - \frac{L^2}{mr^2}}. \quad \blacksquare$$

3) Let W be a finite dimensional vector space over $\mathbf{R}$ or $\mathbf{C}$, and let $U : W \to W$ be a linear map.

(a) Show that U preserves the norm of all vectors if and only if $U^\dagger U = I$. For a real vector space, such a U is called an *orthogonal* map; for a complex vector space, U is called a *unitary* map.

Solution: Proving (if) is easy. Let $y = Ux$, and suppose $U^\dagger U = I$. Then

$$||y|| = \sqrt{\langle y \mid y\rangle} = \sqrt{\langle Ux \mid Ux\rangle} = \sqrt{\left\langle x \mid U^\dagger Ux\right\rangle} = \sqrt{\langle x \mid x\rangle} = ||x||,$$

hence U is norm preserving.

Proving (only if) is slightly trickier. Suppose U is norm preserving. Then the formula

$$\langle x \mid y \rangle = \frac{1}{4}\left[||x+y||^2 - ||x-y||^2\right] + \frac{1}{4i}\left[||x+iy||^2 - ||x-iy||^2\right]$$

tells us that U is also inner product preserving (for a real vector space we only need the first term on the RHS). Hence we have that

$$\left\langle U^\dagger U x \mid y \right\rangle = \langle Ux \mid Uy \rangle = \langle x \mid y \rangle \ \forall\, y \in W.$$

Since this equation holds for all $y \in W$, it tells us that $U^\dagger U x = x$, i.e., $U^\dagger$ is the left inverse of U, as desired. As similar, but more concrete, argument is to use the fact that U maps orthonormal bases to orthonormal bases, then leverage those equations to show that $\left\langle U^\dagger U e_i \mid e_j \right\rangle = \delta_{ij}$, i.e., $U^\dagger U = I$.

Note: Many students wrote on their homework that $\langle x \mid Ax \rangle = \langle x \mid x \rangle \Rightarrow A = I$, for $A = U^\dagger U$. This is *not* true of a general operator A, even if the above equation holds for all x. As an example, let $W = \mathbb{R}^2$ and consider the operator $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then $\langle (x,y) \mid A(x,y) \rangle = x^2 + y^2 = \langle (x,y) \mid (x,y) \rangle$ for an arbitrary vector $(x,y) \in \mathbb{R}^2$. When the above equation holds for all x *and* A is of the form $U^\dagger U$, you can conclude that $A = I$, but that is what you're being asked to show.

Note 2: In the above proof, I showed that $U^\dagger$ is the *left*-inverse of U. Since W is finite dimensional, this actually means that $U^\dagger = U^{-1}$. In an infinite dimensional vector space, it is possible for $U^\dagger$ to be a left inverse without also being the right (and hence actual) inverse. In these cases, $U^\dagger$ will not be norm preserving despite the fact that U is. ■

(b) For an orthogonal or unitary map, U, show that (i) Any eigenvalue, λ, of U satisfies $|\lambda| = 1$. (ii) Any two eigenvectors of U with distinct eigenvalues must be orthogonal. (iii) If a subspace S satisfies $U[S] \subset S$, then $U[S^\perp] \subset S^\perp$ (Hint: Show that $U[S] = S$ and hence $U^{-1}[S] = S$.)

Solution: (i) Let v be an eigenvector with eigenvalue λ. Then

$$\langle v \mid v \rangle = \langle Uv \mid Uv \rangle = \langle \lambda v \mid \lambda v \rangle = |\lambda|^2 \langle v \mid v \rangle \Rightarrow |\lambda| = 1,$$

as desired.

(ii) Let v, w be eigenvectors with distinct eigenvalues λ, μ respectively, and assume $\langle v \mid w \rangle \neq 0$. Then a repetition of the calculation in (i) shows that $\mu^* \lambda = 1 \Rightarrow \arg \mu = \arg \lambda$. Since both eigenvalues are of unit modulus by part (i), this means they are equal, a contradiction. The only resolution is if the equation is trivial, i.e., v and w are orthogonal.

(iii) U is 1-1 by part (i) (it has no zero eigenvalues), so $\dim U[S] = \dim S$. Since we have $U[S] \subseteq S$, this means that $U[S] = S$. Hence any $y \in S$ is of the form $y = Ux, x \in S$. Let $z \in S^\perp$. Then $\langle Uz \mid y \rangle = \langle Uz \mid Ux \rangle = \langle z \mid x \rangle = 0$ since $z \in S^\perp$. Hence $Uz \in S^\perp$, as desired. ■