Physics 31600 — R. Wald
Classical Mechanics — Autumn, 2002

Problem Set IV

Due in class on MONDAY, November 4

1) Consider the "heavy symmetrical top" treated in class (and also discussed in Goldstein and Arnold). Suppose we impose the following additional constraint on the top: The symmetry axis is forced to rotate about the z-axis with uniform angular velocity $\dot{\phi} = \Omega$. Then the top has only two degrees of freedom, which may be "coordinatized" by the remaining Euler angles (θ, ψ).

   (a) Write down the Lagrangian for this system.

   (b) Write down the constants of motion and explicitly reduce the problem of finding the general motion "to quadratures".

   (c) Find the most general choice of initial conditions $(\theta_0, \dot{\theta}_0, \psi_0, \dot{\psi}_0)$ for which the solution has $\dot{\theta} = 0$ for all time (and hence $\theta(t) = \theta_0$). Are these solutions stable insofar as the θ-motion is concerned, i.e., if the initial conditions differ by a small amount from the ones yielding $\theta = 0$, does $\theta(t) - \theta_0$ remain small for all time?

2) Consider a rigid body in "free motion' (i.e. $V = 0$), with a point in the body fixed.

   (a) Show that the motion of uniform rotation about an axis fixed in space is dynamically possible if and only if that axis coincides with a principal axis of the body.

   (b) Suppose that the eigenvalues of the inertia tensor satisfy $I_1 < I_2 < I_3$. Show that the solutions of uniform rotation about the $\hat{e}_1$ and $\hat{e}_3$ axis are stable, but the solution of uniform rotation about the $\hat{e}_2$ axis is unstable. [Note: This problem can be solved by finding the orbits in $\vec{\Omega}$-space by intersecting the surface of constant energy with the surface of constant squared angular momentum, as done in Arnold and Goldstein. Rather than copy their solutions, please do the problem by writing down and solving the (approximate) Euler equations for a small departure from uniform rotation about a principal axis.]

3) Three particles, each of mass m, are constrained to lie on a circle and are connected by identical springs lying on the circle, each of spring constant k, as shown. Find the general solution for the motion of these particles.