Physics 31600 R. Wald
Classical Mechanics Autumn, 2002

Problem Set IV-Solutions

1) Consider the "heavy symmetrical top" treated in class (and also discussed in Goldstein and Arnold). Suppose we impose the following additional constraint on the top: The symmetry axis is forced to rotate about the z-axis with uniform angular velocity $\dot{\phi} = \Omega$. Then the top has only two degrees of freedom, which may be "coordinatized" by the remaining Euler angles (θ, ψ).

(a) Write down the Lagrangian for this system.

(b) Write down the constants of motion and explicitly reduce the problem of finding the general motion "to quadratures".

(c) Find the most general choice of initial conditions $(\theta_0, \dot{\theta}_0, \psi_0, \dot{\psi}_0)$ for which the solution has $\dot{\theta} = 0$ for all time (and hence $\theta(t) = \theta_0$). Are these solutions stable insofar as the θ-motion is concerned, i.e., if the initial conditions differ by a small amount from the ones yielding $\theta = 0$, does $\theta(t) - \theta_0$ remain small for all time?

The Lagrangian for the unconstrained system is:

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta \tag{1}$$

where I_1 and I_3 are the moments of inertia, M is the mass, l is the height of the center of mass along the central principal axis, and ϕ, θ, ψ are the Euler angles. To constrain the system we add a "non-dynamical" field β with the term $-\beta(\dot{\phi} - \Omega)$ in the Lagrangian. This has the effect of replacing $\dot{\phi}$ with Ω in the above Lagrangian. The Lagrangian is , therefore:

$$L = \frac{I_1}{2}(\dot{\theta}^2 + \Omega^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \Omega\cos\theta)^2 - Mgl\cos\theta \tag{2}$$

Since the Lagrangian does not depend on ψ, we can immediately write down one constant of the motion (from the equation for ψ):

$$\frac{d}{dt}[I_3(\dot{\psi} + \Omega\cos\theta)] = \frac{d}{dt}[I_3\omega_3] = 0 \tag{3}$$

Since the system is conservative, the other constant of motion is the energy:

$$E = \frac{I_1}{2}(\dot{\theta}^2 + \Omega^2 \sin^2\theta) + \frac{I_3}{2}\omega_3^2 + Mgl\cos\theta \tag{4}$$

Define $E' = E - \frac{I_3}{2}\omega_3^2$ and $u = \cos\theta$, then we have:

$$E' = \frac{I_1}{2}(\frac{\dot{u}^2}{1-u^2} + \Omega^2(1-u^2)) + Mglu \tag{5}$$

Solving for $\dot{u}$ we have:

$$\dot{u}^2 = (\frac{2(E' - Mglu)}{I_1})(1-u^2) - \Omega^2(1-u^2)^2 \tag{6}$$

Further define $\alpha = \frac{2E'}{I_1}$ and $\beta = \frac{2Mgl}{I_1}$. We can solve for u now with the following quadrature:

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{(\alpha - \beta u)(1-u^2) - \Omega^2(1-u^2)^2}} \tag{7}$$

From the Lagrangian, the equation of motion for θ is

$$I_1\ddot{\theta} = I_1\Omega^2 \sin\theta\cos\theta - I_3(\dot{\psi} + \Omega\cos\theta)(\sin\theta)\Omega + Mgl\sin\theta \tag{8}$$

Setting $\ddot{\theta} = 0$ and noting that $\dot{\psi} + \Omega\cos\theta = \omega_3$ is a constant we see that either $\sin\theta = 0$ or:

$$\cos\theta = \frac{I_3\omega_3\Omega - Mgl}{I_1\Omega^2} \tag{9}$$

Here θ must be a constant, since the right hand side is a constant, thus $\ddot{\theta} = 0 \to \dot{\theta} = 0$. With $\sin\theta(t) = 0$, $\theta(t) = 0$ or π, (although π is unphysical for a real top, we might be dealing with a gyroscope or similar system), then $\dot{\psi} = \omega_3 \mp \Omega$. So, our first sets of allowed initial conditions which produce $\dot{\theta}(t) = 0$ are:

$$\dot{\theta} = 0, \theta = 0, \dot{\psi} = \dot{\psi}_0, \psi = \psi_0$$
$$\dot{\theta} = 0, \theta = \pi, \dot{\psi} = \dot{\psi}_0, \psi = \psi_0$$

We use the equation of motion for θ to determine stability. Expanding about the equilibrium point, to first order in θ this equation is:

$$I_1\ddot{\delta\theta} = [I_1\Omega^2 - I_3(\pm\dot{\psi} + \Omega)\Omega \pm Mgl]\delta\theta \tag{10}$$

where the $\pm$ indicates expansion around $\theta = 0$ or π, respectively. This solution is stable if the term in brackets in negative, otherwise it is unstable. The condition on $\dot{\psi}$ for stability, then, is:

$$\pm\dot{\psi} > [\frac{I_1\Omega^2 \pm Mgl}{I_3\Omega} - \Omega] \tag{11}$$

if $\Omega \neq 0$ and the stability is the same as that of a pendulum if $\Omega = 0$ (as it should be; think about it, you're *forcing* $\Omega = 0$). The other solutions come from Equation (9), which is (substituting for ω_3):

$$\cos\theta_0 = \frac{I_3(\dot{\psi} + \Omega\cos\theta)\Omega - Mgl}{I_1\Omega^2} \tag{12}$$

which gives:

$$\cos\theta_0 = \frac{I_3\Omega\dot{\psi} - Mgl}{(I_1 - I_3)\Omega^2} \tag{13}$$

This has solutions only for the RHS being in the interval $(-1, 1)$. So $\dot{\psi}$ must satisfy:

$$\frac{(I_3 - I_1)\Omega^2 + Mgl}{I_3\Omega} < \dot{\psi} < \frac{(I_1 - I_3)\Omega^2 + Mgl}{I_3\Omega} \tag{14}$$

with $\cos\theta_0$ determined by equation (13), $\dot{\theta}_0 = 0$ and $\psi = \psi_0$. The reverse relationship holds if $I_3 > I_1$. If we again expand the equation of motion for θ around this point we have:

$$I_1\ddot{\delta\theta} = [-I_1\Omega^2\sin^2\theta_0 + I_3\Omega^2\sin^2\theta_0]\delta\theta \tag{15}$$

so these points are stable if $I_1 > I_3$, metastable if $I_1 = I_3$, and unstable otherwise.

2) Consider a rigid body in "free motion' (i.e. $V = 0$), with a point in the body fixed.

(a) Show that the motion of uniform rotation about an axis fixed in space is dynamically possible if and only if that axis coincides with a principal axis of the body.

(b) Suppose that the eigenvalues of the inertia tensor satisfy $I_1 < I_2 < I_3$. Show that the solutions of uniform rotation about the $\hat{e}_1$ and $\hat{e}_3$ axis are stable, but the solution of uniform rotation about the $\hat{e}_2$ axis is unstable. [Note: This problem can be solved by finding the orbits in $\vec{\Omega}$-space by intersecting the surface of constant energy with the surface of constant squared angular momentum, as done in Arnold and Goldstein. Rather than copy their solutions, please do the problem by writing down and solving the (approximate) Euler equations for a small departure from uniform rotation about a principal axis.]

(a) Uniform rotation means we have:

$$\frac{d\vec{L}}{dt})_s = 0 \tag{16}$$

where the s denotes that the coordinate system is space-fixed. This implies:

$$\frac{d\vec{L}}{dt})_s = \frac{d\vec{L}}{dt})_b + \vec{\omega} \times \vec{L} = 0 \tag{17}$$

Uniform motion in the space-fixed frame implies that $\frac{d\vec{L}}{dt})_b = 0$ so we are left with:

$$\vec{\omega} \times \vec{L} = 0 \tag{18}$$

Since we have the relation:

$$\vec{L} = \boldsymbol{I} \cdot \vec{\omega} \tag{19}$$

Equation (18) is satisfied if and only if $\vec{L}$ is parallel to $\vec{\omega}$. This will be true if and only if $\vec{\omega}$ is an eigenvector of $\boldsymbol{I}$. **Q.E.D.**

(b) The Euler equations are:

$$I_i\dot{\omega}_i - \omega_j\omega_k(I_j - I_k) = 0 \tag{20}$$

for i, j, k a cyclic permutation of 1,2,3. Setting $\dot{\omega}_i = 0$ gives (since no 2 I's equal each other):

$$\omega_j\omega_k = 0 \tag{21}$$

which can only be solved if 2 of the 3 ω's are 0. So uniform rotation corresponds to rotation about only 1 principal axis, as above. If we expand the Euler equations about one of these solutions (say for $\omega_1 = \omega_0, \omega_2 = \omega_3 = 0$):

$$I_i\delta\dot{\omega_i} = 0 \tag{22}$$

$$I_j\delta\dot{\omega_j} = (\delta\omega_k)\omega_i(I_k - I_i) \tag{23}$$

$$I_k\delta\dot{\omega_k} = (\delta\omega_j)\omega_i(I_i - I_j) \tag{24}$$

Deferentiating the second two equations with respect to time and substituting the appropriate equations for $\delta\dot{\omega_i}$ we have:

$$I_j\delta\ddot{\omega_j} = \omega_i^2\frac{(I_k - I_i)(I_i - I_j)}{I_k}\delta\omega_j \tag{25}$$

$$I_k\delta\ddot{\omega_k} = \omega_i^2\frac{(I_i - I_j)(I_k - I_i)}{I_j}\delta\omega_k \tag{26}$$

If we take $i = 1$ or $i = 3$ then the right hand coefficients are negative, hence the direction of motion is stable. If $i = 2$, the coefficients are positive and the motion is unstable.

3) Three particles, each of mass m, are constrained to lie on a circle and are connected by identical springs lying on the circle, each of spring constant k, as shown. Find the general solution for the motion of these particles.

If the radius of the circle is a, the Lagrangian for this system is:

$$L = \sum_{i=1}^{3} \frac{1}{2} m a^2 \dot{\phi}_i^{\,2} - \frac{1}{2} k a^2 (\phi_i - \phi_{i+1})^2 \tag{27}$$

We can rewrite this equation as:

$$L = \frac{1}{2} m a^2 \dot{\vec{\phi}}^2 - \frac{1}{2} k a^2 \vec{\phi} \cdot \boldsymbol{A} \vec{\phi} \tag{28}$$

where:

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{29}$$

The equation of motion is:

$$m a^2 \ddot{\vec{\phi}} = -k a^2 \boldsymbol{A} \vec{\phi} \tag{30}$$

The solutions of this equation are $\vec{\phi}(t) = \vec{\phi_i} \exp(-i\omega_i t)$, where $\vec{\phi_i}$ is an eigenvector of $\boldsymbol{A}$ with eigenvalue λ_i. This yields:

$$m a^2 \omega_i^2 = k a^2 \lambda_i \tag{31}$$

so the frequency of oscillation of each mode is:

$$\omega_i = \sqrt{\frac{k \lambda_i}{m}} \tag{32}$$

A quick calculation shows that the eigenvectors are:

$$\vec{\phi_1} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \tag{33}$$

$$\vec{\phi_2} = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \tag{34}$$

$$\vec{\phi_3} = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \tag{35}$$

with eigenvalues

$$\lambda_1 = 0 \tag{36}$$

$$\lambda_2 = 3 \tag{37}$$

$$\lambda_3 = 3 \tag{38}$$

These have $\omega_1 = 0$ and $\omega_2 = \omega_3 = \sqrt{\frac{3k}{m}}$. Since $\omega_1 = 0$, we must mulitply $\vec{\phi_1}$ by a linear function of time (because this is also a solution). The general motion is then:

$$\vec{\phi}(t) = (A_1 + B_1 t)\vec{\phi_1} + (A_2 \vec{\phi_2} + A_3 \vec{\phi_3}) \exp(-i\sqrt{\frac{3k}{m}} t) \tag{39}$$

with initial conditions determining the constants as:

$$\vec{\phi(0)} = A_1\vec{\phi_1} + A_2\vec{\phi_2} + A_3\vec{\phi_3} \tag{40}$$

$$\dot{\vec{\phi(0)}} = B_1\vec{\phi_1} - i\sqrt{\frac{3k}{m}}(A_2\vec{\phi_2} + A_3\vec{\phi_3}) \tag{41}$$