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Classical Mechanics Autumn, 2002

Problem Set V Solutions v1.1

1) A body of mass m is constrained to move only horizontally (i.e., in the x-direction) and is attached to a fixed support by a spring of spring constant k. A particle also of mass m is attached to the first body by a massless rod of length l as shown. A gravitational field is present, so the potential energy of the second particle is mgz. Suppose that, initially, the first body is at the equilibrium position of the spring, but the particle below it is displaced from the vertical by a small angle θ_0 as shown. Solve for the subsequent motion of both bodies.

Solution: Let x be the position of the mass moving horizontally. Then the position of the pendulum mass is $x_p = x + l\sin\theta$, $y_p = -l\cos\theta$. Hence we can express the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + 2l\cos\theta\dot{\theta}\dot{x} + l^2\dot{\theta}^2\right] - \frac{1}{2}kx^2 + lg\cos\theta.$$

Since the disturbance is small we can expand the Lagrangian to second order in the coordinates and velocities. We further change variables to $u = l\theta$ so that both coordinates will have the same dimension, yielding

$$L = \frac{1}{2}m\left[2\dot{x}^2 + 2\dot{u}\dot{x} + \dot{u}^2\right] - \frac{1}{2}kx^2 - \frac{1}{2}\frac{mg}{l}u^2.$$

Thus, we have coupled oscillations with mass matrix A and potential matrix B given by:

$$\frac{A}{m} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad ; \quad \frac{B}{k} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \text{ with } r = \frac{mg}{lk}.$$

In the above, we have divide B by k and A by m to simplify the arithmetic. The normal modes can be found by diagonalizing B with respect to A. Hence we must solve the equation

$$\det\left(\frac{B}{k} - \frac{m\omega^2}{k}\frac{A}{m}\right) = \begin{vmatrix} 1 - 2\frac{m\omega^2}{k} & -\frac{m\omega^2}{k} \\ -\frac{m\omega^2}{k} & r - \frac{m\omega^2}{k} \end{vmatrix} = 0$$

which gives the eigenfrequencies

$$\omega_\pm^2 = \frac{k}{m}\left(\frac{1}{2} + r \pm \frac{1}{2}\sqrt{1+4r^2}\right) = \frac{k}{2m} + \frac{g}{l} \pm \sqrt{\frac{k^2}{4m^2} + \frac{g^2}{l^2}}.$$

The normal modes are given by the kernels of $B - \lambda A$, so we find

$$N_{\pm} = \text{span}\begin{bmatrix} 1 \\ \frac{1}{2r} - 1 \mp \frac{1}{2r}\sqrt{1+4r^2} \end{bmatrix}.$$

The general solution of each normal mode is of course $X_0 \cos\omega t + V_0 \sin\omega t$, but we are given that the initial conditions are pure displacement, i.e., the initial velocity is zero. Hence we know that we have pure cosine solutions:

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = A\begin{bmatrix} 1 \\ \frac{1}{2r} - 1 - \frac{\sqrt{1+4r^2}}{2r} \end{bmatrix}\cos\omega_+ t + B\begin{bmatrix} 1 \\ \frac{1}{2r} - 1 + \frac{\sqrt{1+4r^2}}{2r} \end{bmatrix}\cos\omega_- t.$$

Applying the initial condition

$$\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ l\theta_0 \end{bmatrix} = \begin{bmatrix} A+B \\ (A+B)\left(\frac{1}{2r}-1\right) + (B-A)\frac{\sqrt{1+4r^2}}{2r} \end{bmatrix},$$

we find that

$$B = -A = \frac{l\theta_0 r}{\sqrt{1+4r^2}}.$$

Switching back to the variable θ, the subsequent motion of the two bodies is

$$\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \frac{-\theta_0}{\sqrt{\frac{l^2k^2}{m^2g^2}+4}}\left\{\begin{bmatrix} l \\ \frac{lk}{2mg} - 1 - \sqrt{\frac{l^2k^2}{4m^2g^2}+1} \end{bmatrix}\cos\left(\frac{k}{2m}+\frac{g}{l}+\sqrt{\frac{k^2}{4m^2}+\frac{g^2}{l^2}}\right)^{\frac{1}{2}} t\right.$$

$$\left. - \begin{bmatrix} l \\ \frac{lk}{2mg} - 1 + \sqrt{\frac{l^2k^2}{4m^2g^2}+1} \end{bmatrix}\cos\left(\frac{k}{2m}+\frac{g}{l}-\sqrt{\frac{k^2}{4m^2}+\frac{g^2}{l^2}}\right)^{\frac{1}{2}} t.\right\}$$

2) Consider a "double pendulum' with each pendulum of mass m and length l, and with a vertical gravitational force $-mg\hat{z}$ acting on each mass. At $t = 0$ the double pendulum is vertical, but the top mass is given a small "kick", so that it has "velocity" $\dot{\theta}_{10}$. Solve for the subsequent motion of the masses.

Solution: Let θ be the angle of the top particle with respect to the vertical, and ψ the angle of bottom particle also with respect to the vertical. If we put the origin at the pivot, then the coordinates of the particles are

$$(x_1, z_1) = l(\sin\theta, -\cos\theta) \quad ; \quad (x_2, z_2) = l(\sin\theta + \sin\psi, -\cos\theta - \cos\theta).$$

The Lagrangian $T - V = T - mgz_1 - mgz_2$ can then be written

$$L = \frac{1}{2}ml^2\left[2\dot{\theta}^2 + \dot{\psi}^2 + 2\cos(\theta-\psi)\dot{\theta}\dot{\psi}\right] + 2mgl\cos\theta + mgl\cos\psi.$$

Since the initial velocity is assumed small, we may expand the Lagrangian to second order in the generalized coordinates and velocities to obtain

$$L = \frac{1}{2}ml^2\left[2\dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\psi}\right] - mgl\left[\theta^2 + \frac{1}{2}\psi^2\right].$$

Thus, we have reduced the problem to couple oscillations with mass matrix A and potential matrix B of the form

$$A = ml^2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad ; \quad B = mgl\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The eigenfrequencies will obey the equation

$$\det(B - \omega^2 A) = \begin{vmatrix} 2mgl - 2ml^2\omega^2 & -ml^2\omega^2 \\ -ml^2\omega^2 & mgl - gl^2\omega^2 \end{vmatrix} = 0,$$

which after one does the arithmetic gives rise to

$$l^2\left(\omega^2\right)^2 - 4gl\omega^2 + 2g^2 = 0 \Rightarrow \omega_\pm^2 = \left(2 \pm \sqrt{2}\right)\frac{g}{l}$$

The normal modes are the null vectors of $B - \omega^2 A$. If we divide B and A by mgl to make this equation dimensionless, we find

$$\ker B - \omega_+^2 A = \ker\begin{bmatrix} -2 - 2\sqrt{2} & -2 - \sqrt{2} \\ -2 - \sqrt{2} & -1 - \sqrt{2} \end{bmatrix} = \ker\begin{bmatrix} \sqrt{2} & 1 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}\right\}.$$

To find the normal mode for ω_-, we can either repeat the above procedure, or exploit the fact that normal modes are orthogonal in the inner product defined by A. Following the latter procedure, we find

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} = \left(2 - \sqrt{2}\right)x + \left(1 - \sqrt{2}\right))y = 0,$$

which gives us that

$$\ker B - \omega_-^2 A = \operatorname{span}\left\{\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\right\}.$$

The general solution for a normal mode is $V_0 \sin \omega t + X_0 \cos \omega t$. Since the initial condition is at zero displacement, we know that we have a pure sine solution:

$$\begin{bmatrix} \theta(t) \\ \psi(t) \end{bmatrix} = V_+\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}\sin \omega_+ t + V_-\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\sin \omega_- t.$$

Applying the initial condition

$$\begin{bmatrix} \theta'(t) \\ \psi'(t) \end{bmatrix} = \omega_+ V_+ \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} + \omega_- V_- \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} \dot{\theta}_{10} \\ 0 \end{bmatrix},$$

solving for V_+ and V_- (trivial, since the ψ' component implies that $\omega_+ V_+ = \omega_- V_-$), and substituting in the positive roots of $\omega_\pm^2$ gives

$$\begin{aligned} \begin{bmatrix} \theta(t) \\ \psi(t) \end{bmatrix} &= \frac{\dot{\theta}_{10}}{2} \sqrt{\frac{l}{\left(2+\sqrt{2}\right) g}} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \sin\left[t\sqrt{\left(2+\sqrt{2}\right)\frac{g}{l}}\right] \\ &+ \frac{\dot{\theta}_{10}}{2} \sqrt{\frac{l}{\left(2-\sqrt{2}\right) g}} \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \sin\left[t\sqrt{\left(2-\sqrt{2}\right)\frac{g}{l}}\right]. \quad \blacksquare \end{aligned}$$

3) Consider a particle of mass m moving in a 2-dimensional plane, with a potential, $V = V(r)$, which depends only on the radius r.
(a) Use the symmetry under rotations to reduce the problem to one of one-dimensional motion in r.

Solution: We begin by writing down the Lagrangian in polar coordinates:

$$L = T - V = \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\theta}^2\right] - V(r).$$

This Lagrangian is cyclic in the angular variable θ, so the angular momentum

$$l = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

is conserved. Since the Lagrangian is time independent and of the standard form $T - V$, the total energy $E = T + V$ is also conserved. Substituting the angular momentum into the expression for E, we obtain

$$E = \frac{1}{2} m \dot{r}^2 + \frac{l^2}{2mr^2} + V(r).$$

Hence, we have reduced the problem to an effective 1d problem. $\blacksquare$

(b) Explicitly write down the equation of motion in r and show that a circular orbit exists at every radius at which $dV/dr > 0$.

Solution: Differentiating the expression for E we obtain:

$$\frac{dE}{dt} = 0 = \dot{r}\left[m\ddot{r} - \frac{l^2}{mr^3} + \frac{dV}{dr}\right].$$

For a general solution $\dot{r} \neq 0$, so we identify the terms in brackets as the equation of motion. (We could also have obtained this by computing the EL equation for r, then substituting l into the resulting equation of motion.) At fixed r, l^2 can be set to an arbitrary nonnegative value by suitable choice of $\dot{\theta}(0)$. Hence the circular orbit

$$r(t) = r_0 \quad ; \quad l_0^2 = m r_0^3 \left.\frac{dV}{dr}\right|_{r_0}$$

satisfies the equations of motion. ■

(c) Obtain the general solution for the radial motion resulting from an arbitrary small perturbation about a circular orbit of radius r_0. (Allow the perturbation to change p_ϕ.) Determine the conditions on V such that this circular orbit is stable.

Solution: In parts (a) and (b) we reduced the problem to an effective one-dimensional problem with potential

$$U(r) = \frac{l^2}{2mr^2} + V(r)$$

and also found that the equations of motion for r are

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{dV}{dr} = 0.$$

Either of these can be used as the starting point for our investigation of the effects of perturbations. However, in order to get the equations of motion correct to first order, the potential U must be expanded correctly to second order in the perturbations which is a bit of a mess. We therefore look directly at the equations of motion. Consider a change in the initial conditions from

$$r(0) = r_0 \quad ; \quad \dot{r}(0) = 0 \quad ; \quad \theta(0) = \theta_0 \quad ; \quad \dot{\theta}(0) = \dot{\theta}_0 = \sqrt{\frac{1}{mr_0} \left.\frac{dV}{dr}\right|_{r_0}}$$

by an arbitrary small perturbation:

$$r(0) = r_0 + \delta r_0 \quad ; \quad \dot{r}(0) = \delta\dot{r} \quad ; \quad \theta(0) = \theta_0 + \delta\theta_0 \quad ; \quad \dot{\theta}(0) = \dot{\theta}_0 + \delta\dot{\theta}_0.$$

The new angular momentum to first order in the perturbations is

$$l^2 = m^2(r_0 + \delta r_0)^4(\theta + \delta\dot{\theta})^2 \approx l_0^2 \left(1 + \frac{4\delta r_0}{r_0} + \frac{2\delta\dot{\theta}_0}{\dot{\theta}_0}\right)$$

Expanding the force $\frac{dV}{dr}$ to first order in the perturbations also we find that the equation of motion for δr is

$$m(\ddot{r}_0 + \delta\ddot{r}) - \frac{l_0^2}{m(r_0 + \delta r)^3}\left(1 + \frac{4\delta r_0}{r_0} + \frac{2\delta\dot{\theta}_0}{\dot{\theta}_0}\right) + \left.\frac{dV}{dr}\right|_{r_0} + \left.\frac{d^2V}{dr^2}\right|_{r_0} \delta r = 0.$$

Expanding the $(r_0+\delta r)^{-3}$ to first order and using the fact that $l_0^2 = mr_0^3\frac{dV}{dr}$ as in part (b), we find that to first order the equation is

$$m\delta\ddot{r} + \left[\frac{3}{r_0}\left.\frac{dV}{dr}\right|_{r_0} + \left.\frac{d^2V}{dr^2}\right|_{r_0}\right]\delta r = \left.\frac{dV}{dr}\right|_{r_0}\left[\frac{4\delta r_0}{r_0} + \frac{2\delta\dot{\theta}_0}{\dot{\theta}_0}\right].$$

The solution of this equation with the initial conditions given above is

$$\delta r(t) = A + (\delta r_0 - A)\cos\omega t + \frac{\delta\dot{r}_0}{\omega}\sin\omega t$$

where A and ω are given by

$$\omega = \sqrt{\frac{1}{m}\left[\frac{3}{r_0}\left.\frac{dV}{dr}\right|_{r_0} + \left.\frac{d^2V}{dr^2}\right|_{r_0}\right]} \quad ; \quad A = \left.\frac{dV}{dr}\right|_{r_0}\left[\frac{4\delta r_0}{r_0} + \frac{2\delta\dot{\theta}_0}{\dot{\theta}_0}\right]\left[\frac{3}{r_0}\left.\frac{dV}{dr}\right|_{r_0} + \left.\frac{d^2V}{dr^2}\right|_{r_0}\right]^{-1}.$$

The frequency ω will be real and the corresponding solution truly oscillatory (as opposed to exponentially growing) if

$$\frac{3}{r_0}\left.\frac{dV}{dr}\right|_{r_0} + \left.\frac{d^2V}{dr^2}\right|_{r_0} > 0,$$

hence this is the condition for stability. This makes perfect sense: the orbit is stable if r_0 corresponds to a minimum of the effective potential U. Finally, we can (although the problem didn't ask us to) also find the solution for θ to first order from conservation of angular momentum:

$$\dot{\theta}(t) = \dot{\theta}_0 + \delta\dot{\theta} = \frac{l}{m(r_0+\delta r)^2} = \frac{l_0}{mr_0^2}\left(1 + \frac{2\delta r_0}{r_0} + \frac{\delta\dot{\theta}_0}{\dot{\theta}_0}\right)\left(1 - \frac{2\delta r}{r_0}\right).$$

As $l_0 = mr_0^2\dot{\theta}_0$ the terms zeroth order in the perturbation cancel, and this equation is

$$\delta\dot{\theta}(t) = \dot{\theta}_0\left(\frac{2\,(\delta r_0 - \delta r(t))}{r_0} + \frac{\delta\dot{\theta}_0}{\dot{\theta}_0}\right) = \dot{\theta}_0\left(\frac{\delta l}{l_0} - \frac{2\delta r(t)}{r_0}\right).$$

Since we've already solved for $\delta r(t)$, the right hand side is a well-defined function of time which gives $\delta\dot{\theta}(t)$ explicitly. It can be integrated to give $\delta\theta(t)$. Since this expression doesn't depend on $\delta\theta_0$, the effect of the perturbation in initial angle is merely to change the offset in $\theta(t)$. The perturbation in the angular velocity has a constant offeset plus an oscillation oppositely signed from that of the radial perturbation. This is something which we could have guessed: since l is still conserved, the radial oscillations must be opposite to the the anglular velocity. ■