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Problem Set VI Solutions

1) Consider a space M with coordinates y^μ and define the Poisson-bracket-like map on functions on M by

$$W(f,g) = \sum_{\mu,\nu} W^{\mu\nu} \frac{\partial f}{\partial y^\mu} \frac{\partial g}{\partial y^\nu}$$

where $W^{\mu\nu}$ is any antisymmetric matrix (i.e., $W^{\nu\mu} = -W^{\mu\nu}$) whose components are constant (i.e., independent of y^μ). Show that W satisfies the Jacobi identity, i.e.,

$$W(W(f,g),h) + W(W(h,f),g) + W(W(g,h),f) = 0$$

Solution: Summation will be implied over repeated indices. First:

$$W(W(f,g),h) = W^{\mu\nu} W^{\sigma\rho} \frac{\partial}{\partial y^\mu} (\frac{\partial f}{\partial y^\sigma} \frac{\partial g}{\partial y^\rho}) \frac{\partial h}{\partial y^\nu} \tag{1}$$

So:

$$\begin{aligned} W(W(f,g),h) + W(W(h,f),g) + W(W(g,h),f) = \\ W^{\mu\nu} W^{\sigma\rho} [\frac{\partial}{\partial y^\mu} (\frac{\partial f}{\partial y^\sigma} \frac{\partial g}{\partial y^\rho}) \frac{\partial h}{\partial y^\nu} + \\ \frac{\partial}{\partial y^\mu} (\frac{\partial h}{\partial y^\sigma} \frac{\partial f}{\partial y^\rho}) \frac{\partial g}{\partial y^\nu} + \\ \frac{\partial}{\partial y^\mu} (\frac{\partial g}{\partial y^\sigma} \frac{\partial h}{\partial y^\rho}) \frac{\partial f}{\partial y^\nu}] \end{aligned} \tag{2}$$

Consider the terms only involving the second derivative of f:

$$W^{\mu\nu} W^{\sigma\rho} [\frac{\partial^2 f}{\partial y^\mu \partial y^\sigma} \frac{\partial g}{\partial y^\rho} \frac{\partial h}{\partial y^\nu} + \frac{\partial h}{\partial y^\sigma} \frac{\partial^2 f}{\partial y^\mu \partial y^\rho} \frac{\partial g}{\partial y^\nu}] \tag{3}$$

Because of the antisymmetry of the $W^{\mu\nu}$, we can switch σ and ρ in the first term at the cost of a minus sign (really we are replacing $W^{\sigma\rho}$ with $-W^{\rho\sigma}$ and then relabeling the indices):

$$W^{\mu\nu} W^{\sigma\rho} [-\frac{\partial^2 f}{\partial y^\mu \partial y^\rho} \frac{\partial g}{\partial y^\sigma} \frac{\partial h}{\partial y^\nu} + \frac{\partial h}{\partial y^\sigma} \frac{\partial^2 f}{\partial y^\mu \partial y^\rho} \frac{\partial g}{\partial y^\nu}] \tag{4}$$

The term in brackets is now symmetric in μ and ρ and antisymmetric in σ and ν. Since we have

$$W^{\mu\nu}W^{\sigma\rho} \rightarrow W^{\rho\sigma}W^{\nu\mu} = (-W^{\sigma\rho})(-W^{\mu\nu}) = W^{\mu\nu}W^{\sigma\rho} \tag{5}$$

all the terms involving the second derivative of f must be zero. The same applies to the second derivatives of g and h because those terms involve cyclic permutations of f, g, and h. Therefore

$$W(W(f,g),h) + W(W(h,f),g) + W(W(g,h),f) = 0 \tag{6}$$

QED

2) Let $f(q,p;t)$ and $g(q,p;t)$ be (possibly time-dependent) observables on a $2n$-dimensional phase space with Hamiltonian $H(q,p;t)$.
(a) Suppose that both f and g are constants of motion, i.e., $df/dt = dg/dt = 0$. Show that $\Omega(f,g)$ also is a constant of motion.

Solution: In general, we have:

$$\frac{df}{dt} = \Omega(f,H) + \frac{\partial f}{\partial t} \tag{7}$$

where H is the Hamiltonian. So we wish to calculate:

$$\frac{d\Omega(f,g)}{dt} = \Omega(\Omega(f,g),H) + \frac{\partial \Omega(f,g)}{\partial t} \tag{8}$$

Using the Jacobi Identity, we have:

$$\frac{d\Omega(f,g)}{dt} = -\Omega(\Omega(H,f),g) - \Omega(\Omega(g,H),f) + \frac{\partial \Omega(f,g)}{\partial t} \tag{9}$$

But from Equation 7 and by hypothesis we have:

$$\frac{df}{dt} = \Omega(f,H) + \frac{\partial f}{\partial t} = 0 \Rightarrow \Omega(H,f) = \frac{\partial f}{\partial t} \tag{10}$$

$$\frac{dg}{dt} = \Omega(g,H) + \frac{\partial g}{\partial t} = 0 \Rightarrow \Omega(g,H) = -\frac{\partial g}{\partial t} \tag{11}$$

(Remember that $\Omega(f,g)$ is antisymmetric in f and g). So:

$$\frac{d\Omega(f,g)}{dt} = -\Omega(\frac{\partial f}{\partial t},g) + \Omega(\frac{\partial g}{\partial t},f) + \frac{\partial \Omega(f,g)}{\partial t} \tag{12}$$

However, because Ω is bilinear and antisymmetric in f and g, we have:

$$\frac{d\Omega(f,g)}{dt} = -\frac{\partial}{\partial t}\Omega(f,g) + \frac{\partial \Omega(f,g)}{\partial t} = 0 \tag{13}$$

QED

(b) Suppose that H itself is a constant of motion and $f(q,p;t)$ is a constant of motion. Show that $\frac{\partial^n f}{\partial t^n}$ also is a constant of motion.

Solution: From Equation 7, we have:

$$\frac{d}{dt}\frac{\partial^n f}{\partial t^n} = \Omega(\frac{\partial^n f}{\partial t^n}, H) + \frac{\partial}{\partial t}\frac{\partial^n f}{\partial t^n} \tag{14}$$

$$\frac{dH}{dt} = \Omega(H,H) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{15}$$

So

$$\frac{\partial H}{\partial t} = 0 \tag{16}$$

since H is a constant of the motion. Equation 16 in turn implies:

$$\frac{\partial}{\partial t}\Omega(f,H) = \Omega(\frac{\partial f}{\partial t}, H) + \Omega(f, \frac{\partial H}{\partial t}) = \Omega(\frac{\partial f}{\partial t}, H) \tag{17}$$

Thus:

$$\frac{d}{dt}\frac{\partial^n f}{\partial t^n} = \frac{\partial^n}{\partial t^n}\Omega(f,H) + \frac{\partial^n}{\partial t^n}\frac{\partial f}{\partial t} = \frac{\partial^n}{\partial t^n}\frac{df}{dt} = 0 \tag{18}$$

QED

3) Consider a particle of mass m in ordinary 3-dimensional space. Let L_x, L_y, L_z denote the usual Cartesian components of angular momentum of the particle, viewed as functions on its 6-dimensional phase space.
(a) Show that $\Omega(L_x, L_y) = L_z$. (Thus, according to problem 2, if L_x and L_y are constants of motion, then so is L_z.)

Solution: The angular momenta are defined by:

$$L_i = (\vec{r} \times \vec{p})_i = x_j p_k - x_k p_j \tag{19}$$

for i, j, and k cyclic permutations of x, y, and z. So

$$\Omega(L_x, L_y) = \sum_i \frac{\partial L_x}{\partial x_i}\frac{\partial L_y}{\partial p_i} - \frac{\partial L_x}{\partial p_i}\frac{\partial L_y}{\partial x_i} \tag{20}$$

From Equation 19, the L_i do not contain either x_i or p_i, therefore:

$$\Omega(L_x, L_y) = \frac{\partial L_x}{\partial z}\frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z}\frac{\partial L_y}{\partial z} = (-p_y)(-x) - (y)(p_x) = L_z \tag{21}$$

QED

(b) Suppose that an observable $f(\vec{x}, \vec{p})$ depends on $\vec{x}$ and $\vec{p}$ only in the scalar combinations $\vec{x} \cdot \vec{x}$, $\vec{x} \cdot \vec{p}$, and $\vec{p} \cdot \vec{p}$, i.e., suppose that f can be written in the form

$$f(\vec{x}, \vec{p}) = h(\vec{x} \cdot \vec{x}, \vec{x} \cdot \vec{p}, \vec{p} \cdot \vec{p})$$

for some function h. Show that $\Omega(f, L_i) = 0$.

Solution: The easy way to do this problem is to note that the L_i are the generating functions of rotations about the x_i axis. Then we have:

$$\frac{df(\vec{x}, \vec{p})}{d\theta_i} = \Omega(f(\vec{x}, \vec{p}), L_i) \tag{22}$$

where θ_i is the angle of rotation about the x_i axis. From Problem 1 of Homework II we know that if we apply a rotation to the configuration variables we must apply the same rotation to the momentum variables. Therefore the combinations $\vec{x} \cdot \vec{x}$, $\vec{x} \cdot \vec{p}$, and $\vec{p} \cdot \vec{p}$ are all invariant under rotations about any axis. This implies that

$$\Omega(f(\vec{x}, \vec{p}), L_i) = \frac{df(\vec{x}, \vec{p})}{d\theta_i} = 0 \tag{23}$$

QED