Physics 31600 R. Wald
Classical Mechanics Autumn, 2002

Problem Set VII

Due in class on Tuesday, December 3

1) (a) Show that the transformation on $2n$-dimensional phase space associated with a coordinate transformation on configuration space, namely:

$$q_i \to Q_i(q)$$

$$p_i \to P_i(q,p) = \sum_j p_j \frac{\partial q_j}{\partial Q_i}$$

is a canonical transformation.

(b) On a 2-dimensional phase space, show that the transformation

$$q \to Q = \ln[\frac{1}{q} \sin p]$$

$$p \to P = q \cot p$$

is canonical.

2) Give an "elementary" proof of Liouville's theorem as follows: Introduce local canonical coordinates $(q_1, p_1, q_2, p_2, ..., q_n, p_n)$ and pretend that these are Cartesian coordinates in $2n$-dimensional Euclidean space, $\mathbf{R}^{2n}$. Consider a bounded region, $\mathcal{R}$, of phase space covered by these coordinates. Let $\mathcal{R}_t$ denote the image of $\mathcal{R}$ under dynamical evolution by time t. Argue that the volume, $V(\mathcal{R}_t)$, of $\mathcal{R}_t$ must satisfy

$$\frac{dV}{dt} = \int_{\partial \mathcal{R}_t} \vec{h} \cdot \hat{n} dS$$

where $\partial \mathcal{R}_t$ denotes the boundary of $\mathcal{R}_t$, $\vec{h}$ denotes the Hamiltonian vector field, and $\hat{n}$ denotes the unit outward pointing normal to $\partial \mathcal{R}_t$. Then show that $dV/dt = 0$.

3) (a) Find the generating function, $F_2(q,P)$, for the canonical transformation of problem 1(a) above.

(b) Find the generating functions, $F_1(q,Q)$ and $F_2(q,P)$, for the canonical transformation of problem 1(b) above.

(c) Explicitly choose a function of two variables, $f(x,y)$. Then obtain the canonical transformations on a 2-dimensional phase space that it generates via the generating functions (i) $F_1(q,Q) = f(q,Q)$ and $F_2(q,P) = f(q,P)$.