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Problem Set VII Solutions

1) (a) Show that the transformation on $2n$-dimensional phase space associated with a coordinate transformation on configuration space, namely:

$$q_i \to Q_i(q)$$
$$p_i \to P_i(q,p) = \sum_j p_j \frac{\partial q_j}{\partial Q_i}$$

is a canonical transformation.

Solution: As was shown in class, in order to show that a transformation is canonical it is sufficient to demonstrate that the new coordinates satisfy the Poisson bracket as functions of the old coordinates. We first compute the easy bracket:

$$\Omega^{(p,q)}\left(Q_i, Q_j\right) = \sum_{k=1}^{n} \left(\frac{\partial Q_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial Q_j}{\partial q_k} \right) = 0$$

since the Q_i are independent of p_i. Next up are the mixed PQ brackets:

$$\Omega^{(p,q)}\left(Q_i, P_j\right) = \sum_{k=1}^{n} \left(\frac{\partial Q_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} \right) = \sum_{k=1}^{n} \frac{\partial Q_i}{\partial q_k} \frac{\partial q_k}{\partial Q_j} - 0 = \frac{\partial Q_i}{\partial Q_j} = \delta_{ij},$$

as desired. The last equality follows from the fact that the Q are good coordinates on configuration space so they must obey that relation. Finally, we need to do the P brackets, which are bit of mess:

$$\Omega^{(p,q)}\left(P_i, P_j\right) = \sum_{k=1}^{n} \left(\frac{\partial P_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial P_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} \right) = \sum_{k=1}^{n} \left(\left(\sum_{l=1}^{n} p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_i} \right) \frac{\partial q_k}{\partial Q_j} - \frac{\partial q_k}{\partial Q_i} \left(\sum_{l=1}^{n} p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_j} \right) \right).$$

Notice that in the terms involving second partial derivatives, we cannot blithely interchange order of differentiation because different things are being held fixed. For the q derivatives, the other $q's$ are being held fixed, while for the Q derivative it is the other $Q's$. Now we interchange the order of summation and note that

$$\sum_{k=1}^{n} \frac{\partial^2 q_l}{\partial q_k \partial Q_i} \frac{\partial q_k}{\partial Q_j} = \frac{\partial^2 q_l}{\partial Q_j \partial Q_i}.$$

Since these are partial derivatives where the same variables are being held fixed, we can just interchange order of differentiation to yield

$$\sum_{l=1}^{n} p_l \left(\frac{\partial^2 q_l}{\partial Q_j \partial Q_i} - \frac{\partial^2 q_l}{\partial Q_i \partial Q_j} \right) = 0,$$

as desired. ■

(b) On a 2-dimensional phase space, show that the transformation

$$q \to Q = \ln[\frac{1}{q} \sin p]$$

$$p \to P = q \cot p$$

is canonical.

Solution: Again, we simply need to check that the new coordinates satisfy the canonical commutation relations as functions of the old coordinates. The QQ and PP brackets are trivial because the Poisson bracket is antisymmetric and so vanishes for any function of p and q. Thus, we just need to check the QP bracket, which is

$$\Omega^{(p,q)}(Q, P) = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \left[\frac{-1}{q}\right]\left[q(-\csc^2 p)\right] - \left[\frac{\cos p}{\sin p}\right][\cot q] = \frac{1 - \cos^2 p}{\sin^2 p} = 1,$$

as desired. ■

2) Give an "elementary" proof of Liouville's theorem as follows: Introduce local canonical coordinates $(q_1, p_1, q_2, p_2, ..., q_n, p_n)$ and pretend that these are Cartesian coordinates in $2n$-dimensional Euclidean space, $\mathbf{R}^{2n}$. Consider a bounded region, $\mathcal{R}$, of phase space covered by these coordinates. Let $\mathcal{R}_t$ denote the image of $\mathcal{R}$ under dynamical evolution by time t. Argue that the volume, $V(\mathcal{R}_t)$, of $\mathcal{R}_t$ must satisfy

$$\frac{dV}{dt} = \int_{\partial \mathcal{R}_t} \vec{h} \cdot \hat{n} dS$$

where $\partial \mathcal{R}_t$ denotes the boundary of $\mathcal{R}_t$, $\vec{h}$ denotes the Hamiltonian vector field, and $\hat{n}$ denotes the unit outward pointing normal to $\partial \mathcal{R}_t$. Then show that $dV/dt = 0$.

Solution: Consider a volume $\mathcal{R}_t$ at a particular time in phase space. Corresponding to this volume we can construct an ensemble of particles wherein each point in phase space $(q_1, p_1, q_2, p_2, ..., q_n, p_n)$ contained within the volume corresponds to the state of precisely one particle in the ensemble. In this case the volume is equal to the number of particles in the ensemble. If we wish to know the rate of change of the number of particles within this volume at time t, we must calculate the flux of the particles across the boundary of the volume of phase space. That is:

$$\frac{dV}{dt} = \int_{\partial \mathcal{R}_t} \dot{\vec{x}} \cdot \hat{n} dS \tag{1}$$

where $\vec{x}$ is the coordinate vector *in phase space.*

But, Hamilton's equations of motion are precisely $\dot{\vec{x}} = \vec{h}$, so we have the result:

$$\frac{dV}{dt} = \int_{\partial \mathcal{R}_t} \vec{h} \cdot \hat{n} dS \tag{2}$$

Furthermore, the divergence theorem allows us to write this as:

$$\frac{dV}{dt} = \int_{\partial\mathcal{R}_t} \vec{h}\cdot\hat{n}dS = \int_{\mathcal{R}_t} \nabla\cdot\vec{h}d\vec{x} \tag{3}$$

However, we have:

$$\nabla\cdot\vec{h} = \sum_i \frac{\partial}{\partial q_i}\frac{\partial\mathcal{H}}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial\mathcal{H}}{\partial q_i} = 0 \tag{4}$$

Therefore $\frac{dV}{dt} = 0$

3) (a) Find the generating function, $F_2(q, P)$, for the canonical transformation of problem 1(a) above.

Solution: F_2 is given by $F_2 = F_1 + \sum_j P_jQ_j$. Thus, we first need to find F_1 and then we can simply compute F_2. Now F_1 satisfies

$$dF_1 = \sum_j (p_jdq_j - P_jdQ_j) = \sum_j \left[p_jdq_j - \left(\sum_k p_k\left(\frac{\partial q_k}{\partial Q_j}\right)_{\{Q,p\}}\right)\left(\sum_l \left(\frac{\partial Q_j}{\partial q_i}\right)_{\{q,p\}} dq_i\right)\right]$$

For fixed k and i, we have that

$$\sum_j \left(\frac{\partial Q_j(q)}{\partial q_i}\right)_{\{q,p\}}\left(\frac{\partial q_k(q)}{\partial Q_j}\right)_{\{Q,p\}} = \left(\frac{\partial q_k}{\partial q_i}\right)_{\{q,p\}} = \delta_{ik},$$

so that we get

$$dF_1 = \sum_j [p_jdq_j] - \sum_{k,l} [p_k\delta_{ik}dq_i] = \sum_j [p_jdq_j] - \sum_k [p_kdq_k] = 0.$$

Thus, we can take $F_1(q, Q) = 0$. Why can we take the generating function to be constant (which means that p and P are equal to zero)? Because q and Q do not form good coordinates on phase space; they only cover configuration space. Thus, we have the simple solution

$$F_2(q, P) = \sum_j Q_j(q)P_j,$$

where we must explicitly insert the expression for Q_j in terms of q. ■

(b) Find the generating functions, $F_1(q, Q)$ and $F_2(q, P)$, for the canonical transformation of problem 1(b) above.

Solution: Again, we start we the formula

$$dF_1(q, Q) = pdq - PdQ.$$

Since we want F_1 in terms of Q and q, we solve for p and P in terms of Q and q:

$$p = \sin^{-1}\left(qe^Q\right) \quad ; \quad P = q\cot p = \frac{\sqrt{1-q^2e^{2Q}}}{e^{2Q}}$$

so that

$$dF_1 = \sin^{-1}\left(qe^Q\right) dq - \frac{\sqrt{1-q^2e^{2Q}}}{e^{2Q}} dQ.$$

Integrating the coefficient of dq (which is $\frac{\partial F_1}{\partial q}$) with respect to q we get

$$F_1 = q\sin^{-1}\left(qe^Q\right) + \frac{\sqrt{1-q^2e^{2Q}}}{e^Q} + f(Q).$$

Simply differentiating the above with respect to Q and comparing with the coefficient of dQ shows we can take $f(Q) = 0$, so we find that

$$F_1 = q\sin^{-1}\left(qe^Q\right) + \frac{\sqrt{1-q^2e^{2Q}}}{e^Q}.$$

Now we know that $F_2(q,P) = F_1(q, Q(q,P)) + PQ(q,P)$, so all we need to do is solve for Q in terms of q, P and substitute into the above expression. Using the above expression for P, we find that

$$P = \frac{\sqrt{1-q^2e^{2Q}}}{e^Q} \quad \Rightarrow \quad Q = -\frac{1}{2}\ln\left(P^2+q^2\right)$$

which gives that

$$F_2(q,P) = q\sin^{-1}\left[\frac{q}{\sqrt{P^2+q^2}}\right] + P - \frac{P}{2}\ln\left(P^2+q^2\right) = q\cot^{-1}\frac{P}{q} + P - \frac{P}{2}\ln(P^2+q^2) \quad \blacksquare$$

(c) Explicitly choose a function of two variables, $f(x,y)$. Then obtain the canonical transformations on a 2-dimensional phase space that it generates via the generating functions (i) $F_1(q,Q) = f(q,Q)$ and $F_2(q,P) = f(q,P)$.

Solution: Clearly, your results will depend on what function you choose. I will demonstrate with two different examples. I will start with my favourite function on $\mathbb{R}^2$, namely $f(x,y) = x + y$. Then

$$F_1(q,Q) = q + Q \quad \Rightarrow \quad p = \frac{\partial F_1}{\partial q} = 1 \quad ; \quad P = -\frac{\partial F_1}{\partial Q} = -1$$

$$F_2(q,P) = q + P \quad \Rightarrow \quad p = \frac{\partial F_1}{\partial q} = 1 \quad ; \quad Q = \frac{\partial F_1}{\partial P} = 1.$$

This illustrates an important point: even though any function will generate a "canonical transformation", there is no guarantee that the resulting functions will be good coordinates on phase space even if the generating function satisfies every nice property of which you can think. Next up is slightly more complicated function, $f(x,y) = (x+y)^2$. Then

$$F_1(q,Q) = q^2 + +2Qq + Q^2 \quad \Rightarrow \quad p = \frac{\partial F_1}{\partial q} = 2(Q+q) \quad ; \quad P = -\frac{\partial F_1}{\partial Q} = 2(Q+q).$$

Solving for P and Q in terms of p and q gives $q = Q$ and $p = P$. This is nothing but the identity transformation. For F_2 we get

$$F_2(q,P) = q^2 + 2qP + P^2 \quad \Rightarrow \quad p = \frac{\partial F_1}{\partial q} = 2(q+P) \quad ; \quad Q = \frac{\partial F_1}{\partial P} = 2(q+P).$$

Solving for P and Q again gives $Q = p$ and $P = p - 2q$. So this transformation turns the momenta into the coordinates and then does a little rearranging on the new momenta. ■