Physics 35200 R. Wald
Statistical Mechanics Spring, 2002

Problem Set I

Due in class on Monday, April 8

1) (Counts as 2 problems) Consider the "unit torus", T, obtained by identifying the opposite sides of a unit square. Consider the dynamics on the torus defined by motion with unit velocity along a straight line with slope α, as considered in class. Restrict attention to the case where α is *irrational*.

   a) Show that any bounded function $f : T \to \mathbf{R}$ which is invariant under dynamical evolution (i.e., such that $f(x_t) = f(x)$ for all x) must be constant "almost everywhere" (i.e., except possibly on a set of measure zero). [Hint: Expand f in a double Fourier series and examine the "time evolution" behavior of its Fourier coefficients. You may use the fact that any square-integrable function on T (and, thus, in particular, any bounded function) is uniquely determined "almost everywhere" by its Fourier coefficients.] Note that this proves that the dynamics on T is metric indecomposible: If T could be decomposed into disjoint sets of nonzero measure, A and B, which are each invariant under time evolution, the function f defined by $f(x) = 0$ for all x in A and $f(x) = 1$ for all x in B would be a non-constant function invariant under dynamical evolution.

   b) Now let $h : T \to \mathbf{R}$ be any function on T (not necessarily invariant under dynamical evolution) such that the double Fourier series of h converges absolutely and uniformly to h (so that interchanges of summation and integration can be justified; this holds automatically if h is C^3). Show explicitly that the conclusion of Birkhoff's theorem holds: the "time average" of h along *any* orbit is equal to the "phase average" of h.

   c) Show that the dynamics defined above is *not* mixing.

2) Consider a 1 m^3 box of gas at ordinary room temperature conditions ($n \sim 10^{19}$ particles/cm^3, $v \sim 10^4$ cm/sec, collision mean free time $\tau \sim 10^{-9}$ sec). Estimate the amount of time it would take for the gas to return to a configuration where each particle is within 1 cm^3 of its initial position and has velocity components within 10% of its initial velocity components. (Don't be afraid to make very crude estimates but please explain your assumptions and methods.)