Physics 35200 R. Wald
Statistical Mechanics Spring, 2002

Problem Set I Solutions

1) (Counts as 2 problems) Consider the "unit torus", T, obtained by identifying the opposite sides of a unit square. Consider the dynamics on the torus defined by motion with unit velocity along a straight line with slope α, as considered in class. Restrict attention to the case where α is *irrational.*

a) Show that any bounded function $f : T \to \mathbf{R}$ which is invariant under dynamical evolution (i.e., such that $f(x_t) = f(x)$ for all x) must be constant "almost everywhere" (i.e., except possibly on a set of measure zero). [Hint: Expand f in a double Fourier series and examine the "time evolution" behavior of its Fourier coefficients. You may use the fact that any square-integrable function on T (and, thus, in particular, any bounded function) is uniquely determined "almost everywhere" by its Fourier coefficients.] Note that this proves that the dynamics on T is metric indecomposable: If T could be decomposed into disjoint sets of nonzero measure, A and B, which are each invariant under time evolution, the function f defined by $f(x) = 0$ for all x in A and $f(x) = 1$ for all x in B would be a non-constant function invariant under dynamical evolution.

Solution: Following the hint, we begin by writing $f(x,y) = \sum_{k,l} e^{2\pi i(kx+ly)} \tilde{f}_{k,l}$. The Fourier inversion formula gives $\tilde{f}_{k,l} = \int_{x=0}^{1} \int_{y=0}^{1} e^{-2\pi i(kx+ly)} f(x,y)$. Now consider $f_t(x,y) = f\left(x + \frac{t}{\sqrt{1+\alpha^2}}, y + \frac{t\alpha}{\sqrt{1+\alpha^2}}\right)$. Its Fourier coefficients are defined by the same formula, and they must be equal to the Fourier coefficients of f since it is an invariant function. Thus, we obtain

$$0 = \tilde{f}_{k,l} - (\tilde{f}_t)_{k,l} = \int_{x=0}^{1} \int_{y=0}^{1} e^{-2\pi i(kx+ly)} \left(f(x,y) - f_t(x,y)\right)$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} e^{-2\pi i(kx+ly)} \left(1 - e^{2\pi i t\left(k\frac{1}{\sqrt{1+\alpha^2}} + l\frac{\alpha}{\sqrt{1+\alpha^2}}\right)}\right) f(x,y)$$

using a change of variable in f_t. Since α is irrational, we cannot have $k = -\alpha l$ unless k and l are both zero, so the phase in the second exponential on the last line is non-zero. Thus, except for $k = l = 0$, the middle term in parentheses will be some non-zero constant (i.e., independent of x and y) for general values of t. This means that the difference between the Fourier coefficients will only vanish if the Fourier coefficients themselves vanish. Hence $f(x,y) = \tilde{f}_{0,0}$ a.e., and f is constant almost everywhere.

b) Now let $h : T \to \mathbf{R}$ be any function on T (not necessarily invariant under dynamical evolution) such that the double Fourier series of h converges absolutely and uniformly to h (so that interchanges of summation and integration can be justified; this holds automatically if h is C^3). Show explicitly that the conclusion of Birkhoff's theorem holds: the "time average" of h along *any* orbit is equal to the "phase average" of h.

Solution: Again, we write $h = \sum_{k,l} e^{2\pi i(kx+ly)}\tilde{h}_{k,l}$. The phase average of h is

$$\int_T \sum_{k,l} e^{2\pi i(kx+ly)}\tilde{h}_{k,l} = \sum_{k,l}\int_T e^{2\pi i(kx+ly)}\tilde{h}_{k,l} = \tilde{h}_{0,0}$$

(which can be seen from either the explicit integral or the orthogonality of Fourier functions). On the other hand, the time average is

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt h(x(t),y(t)) = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \sum_{k,l} e^{2\pi i\left(kx+ly+t\left(\frac{k}{\sqrt{1+\alpha^2}}+\frac{l\alpha}{\sqrt{1+\alpha^2}}\right)\right)}\tilde{h}_{k,l}$$

$$= \sum_{k,l}\lim_{T\to\infty}\frac{1}{T}\left(\frac{k}{\sqrt{1+\alpha^2}}+\frac{l\alpha}{\sqrt{1+\alpha^2}}\right)^{-1}\left[e^{2\pi i\left(kx+ly+T\left(\frac{k}{\sqrt{1+\alpha^2}}+\frac{l\alpha}{\sqrt{1+\alpha^2}}\right)\right)} - e^{2\pi i(kx+ly)}\right]\tilde{h}_{k,l}.$$

Since α is irrational, the above formula makes sense for all k, l except $k = l = 0$. In that case, we have the time average of a constant, so we simply get back $\tilde{h}_{0,0}$. For all other k, l we note that the term in brackets has modulus bounded by 2. Thus, as $T \to \infty$, each term goes to zero. Thus, the time average is equal to $\tilde{h}_{0,0}$, which is the same as the phase average, as desired.

c) Show that the dynamics defined above is *not* mixing.

Solution: We can solve this problem by directly considering the definition. It says that for any two sets A and B,

$$\lim_{t\to\infty}\mu(A_t\cap B) = \mu(B)\frac{\mu(A)}{\mu(T)}.$$

So, all that we need to do is produce two sets which fail that criterion. Take, for example, $A = B = \{(x,y)|\ 0 \le x \le 0.5\}$, i.e., the left half of the box. Every $\sqrt{1+\alpha^2}/2$ seconds, the orbit A_t will lie entirely in the right half of the torus, so that $A_t \cap B = \emptyset$. Therefore $\mu(A_t\cap B)$ can never asymptotically reach the limiting value of $1/2$, and the flow is not mixing.

Note: The key idea of this proof was that the time evolution preserves the "shape" of the set, hence one set could never spread out to evenly cover another set. In the language of differential geometry, this is a result of the Hamiltonian vector field being a Killing field, so that the phase flow is a family of isometries.

2) Consider a 1 m^3 box of gas at ordinary room temperature conditions ($n \sim 10^{19}$ particles/cm^3, $v \sim 10^4$ cm/sec, collision mean free time $\tau \sim 10^{-9}$ sec).

Estimate the amount of time it would take for the gas to return to a configuration where each particle is within 1 cm^3 of its initial position and has velocity components within 10% of its initial velocity components. (Don't be afraid to make very crude estimates but please explain your assumptions and methods.)

Solution: We begin by considering a single particle. There are two time scales in this problem: the time it takes a particle to "forget" its momentum, and the time it takes it to "forget" its position. Each time it collides with another particle, its momentum is, in effect, completely randomized, so the first length scale is the mean collsion time 10^{-9} seconds. The second time scale can be taken to be the length of the box divided by the mean velocity, or 10^{-2} seconds (this can be computed more accurately using a random walk, but the factor introduced will be negligible for our purposes). Since one time scale is so much longer than the other, we can ignore the momenta, and simply consider "taking snapshots" every 0.01 seconds. As we want the particle to be within a 1% of its original position in each of the 3 axes, there is a 10^{-6} probability that the particle will be there in each snapshot. Therefore, the probability that all 10^{25} particles will be back in there original positions is $10^{-6\times 10^{25}}$! As we're taking snap shots once every 0.01 seconds, the expected time to reach this configution is $10^{\left(6\times 10^{25}\right)-2}$ seconds, or a really really really really really long time.