Dynamics in Stationary, Non-Globally Hyperbolic Spacetimes

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The Bottom Line

- There always are local solutions to the Klein-Gordon equation.
- In globally hyperbolic spacetimes, there exist global solutions with a host of important properties (below).
- The present work establishes the existence of global solutions to the wave equation in causal, stationary, non-globally hyperbolic spacetimes.
- Further, there is a prescription for assigning solutions to initial data which preserves important properties of the well-posed problem.
(Non-)Global Hyperbolicity

- The domain of dependence $D(\Sigma_0)$ is the set of points $p$ such that every inextendible timelike curve through $p$ intersects $\Sigma_0$.
- Globally hyperbolic spacetimes $M$ have a Cauchy surface $\Sigma_0$ (for which $D(\Sigma_0) = M$).
Well-posedness

- Global-hyperbolicity guarantees the **well-posedness** of initial value problem for scalar test fields:
  - there is a unique solution throughout spacetime for given initial data,
  - solutions depend continuously on initial data, and
  - smooth initial data produce smooth solutions.

- In stationary spacetimes, solutions also conserve energy.
Non-Globally-Hyperbolic Spacetimes

- In general, non-globally hyperbolic spacetimes have an ill-posed initial value problem.
- The present work shows that a prescription exists in a large class of general stationary (not necessarily static) spacetimes.
Stationary Spacetimes

- $(M, g_{ab})$ is **stationary** if it has an everywhere timelike, complete Killing vector field $t^a$.
- Black hole solutions are **not** stationary.
- A **static** spacetime has time-reversal symmetry as well.
Plan of Attack

The general plan is as follows:

1. Construct a suitable Hilbert space of initial data.
2. Convert the PDE problem into a Hilbert space problem.
4. Convert back and show that the result is a sensible PDE solution.
The energy Hilbert space $\mathcal{H}_A$ is the completion of $C^\infty_0(\Sigma) \oplus C^\infty_0(\Sigma)$ in the inner product

$$\langle \Phi | \Phi \rangle := \int_\Sigma d\gamma T_{ab} n^a t^b.$$
Lapse and Shift

Recall that the lapse function $\alpha$ and shift-vector $\beta^a$ are defined by

$$t^a = \alpha n^a + \beta^a,$$

where $\beta^a n_a = 0$. Note that $-t^a t_a = \alpha^2 - \beta^2$. 

$x=0$

$\beta$

$\alpha n^a$

$\beta$

$\beta$

$t=0$
The Klein-Gordon Equation

- The Klein-Gordon equation is a second order hyperbolic differential equation:
  \[(\nabla^a \nabla_a - m^2)\varphi = 0.\]

- Using the canonical momentum \(\pi = n^a \nabla_a \varphi\), and letting \(\Phi = (\varphi, \pi)\), this equation may be rewritten as a first order system:
  \[\frac{\partial}{\partial t} \Phi = -h\Phi.\]
Properties of $h$

- $h$'s explicit form is

$$-h = \begin{bmatrix} \beta^a D_a & \alpha \pi \\ D^a(\alpha D_a) - \alpha m^2 & -(D_a \beta^a) - \beta^a D_a \end{bmatrix}$$

- $h$ is a $2 \times 2$ matrix operator containing only spatial derivatives.
- The form of $h$ depends on the choice of slicing.
- $h$, acting on $C^\infty_0(\Sigma) \oplus C^\infty_0(\Sigma)$, is anti-Hermitian in the energy inner product.
Assumptions

- Restrict attention to fields with

\[ m^2 > 0. \quad \text{(PosMass)} \]

- It is necessary that the slicing obey

\[ \alpha - \frac{\beta_a \beta^a}{\alpha} \geq \epsilon > 0. \quad \text{(NonNull)} \]

This implies that \( \alpha \geq \epsilon \) and \( \alpha^2 - \beta^2 \geq \epsilon^2 \).
The Prescription(s)

1. Start with spacetime possessing slicings which obey (NonNull).
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2. Choose any such slicing and construct the space $\mathcal{H}_A$. 

Recall that $\partial_t \Phi(t, x) = -h \Phi(t, x)$. Choose a skew-adjoint extension $h_{SA}$ of $h$ and use the spectral theorem to define $\Phi_t(x) = e^{-h_{SA}t} \Phi_0(x)$. Notice that $\Phi_t$ is defined at every point of space, and the transformation from $\Phi_0$ to $\Phi_t$ is unitary.
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3. Define $h$ as above on $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$.

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The Prescription(s)

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   Recall that $\frac{\partial}{\partial t} \Phi(t, x) = -h\Phi(t, x)$.
4. Choose a skew-adjoint extension $h^{SA}$ of $h$ and use the spectral theorem to define
   
   $\Phi_t(x) = e^{-h^{SA}t} \Phi_0(x)$. 
Existence of Extension

Theorem I

Let \((M, g_{ab})\) be a stationary spacetime, and consider a minimally coupled Klein-Gordon equation subject to (PosMass). If \((\Sigma_t, \gamma_{ab})\) is a foliation of satisfying (NonNull), then \(h\) possesses at least one skew-adjoint extension. Further, this extension \(h^I\) is invertible.
Properties of Solutions

Theorem II

Assume the conditions of Theorem I hold. Let $\Phi_0$ be smooth data of compact support. If $\Phi$ is the solution defined via the prescription for any $h^{SA}$ and $\Psi$ the maximal Cauchy evolution of $\Phi_0$, then

(a) $\Phi = \Psi$ within $D(\Sigma_0)$,

(b) $\Phi$ varies continuously with initial data,

(c) smooth data of compact support give rise to smooth solutions, and

(d) $\Phi$ conserves energy.
The Static Case

Theorem III

Let \((M, g_{ab})\) be a static spacetime obeying (NonNull) in the static slicing. If (PosMass) holds, then \(h\) is essentially skew-adjoint. Further, the stationary spacetime prescription agrees with a definite prescription in the Wald-Ishibashi formalism for static spacetimes.
Conclusions

- A non-empty class of prescriptions for defining dynamics can be given in stationary spacetimes obeying the mild condition (NonNull).
- Any prescription in this class automatically conserves energy.
- In the static case, there is only one prescription in the class. It corresponds to a definite prescription in Wald’s formalism.
- As an added bonus, linear field quantization is possible.
Open Questions

- Is the extension $h^l$ unique?
- How do the classes in different slicings compare?

In the static case, this formalism can be modified to include all Wald-Ishibashi dynamics. Is something similar true in the general case?