Investigating the Relationship Between Cavendish Temperature Fluctuation and Torsional Oscillation

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Abstract

The Cavendish apparatus measures the gravitational attraction between lead weights and a torsional pendulum to determine the value of G. The temperature near the apparatus was collected along with torsional pendulum's position data. The calculated value for the gravitational constant was $7.03 \cdot 10^{-11} \pm 0.15 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$. *The period of the temperature oscillation was* 2499.9 ± 0.05 s, *and this fluctuation causes changes in the position data.*

Introduction

In 1797, Henry Cavendish constructed an apparatus to measure the minuscule gravitational attraction between two masses. In February 2010, Paul Riggins and I replicated the experiment and discovered results of equilibrium position and period varied greatly over time. The oscillations did not approach zero as the model predicted, but instead reached a vaguely sinusoidal state. To conserve of energy, there must have been some energy input to the system. This experiment tests the hypothesis that the air conditioning cycles in the room perturbed the system.

Theory

A dumbbell suspended on a torsional fiber and displaced a small angle θ from equilibrium position will oscillate back and forth. The fiber acts as a torsional spring, governed by the equation

$$\tau_{band} = -\kappa\theta \tag{1}$$

where θ is the angular displacement from equilibrium and κ is the torsion constant of the fiber. Two large lead weights of mass m_1 attract the dumbbell mass m_2 with the force given by the law of universal gravitation,

$$F = \frac{Gm_1m_2}{b^2} - \frac{Gm_1m_2}{4d^2+b^2} sin\phi = \frac{Gm_1m_2}{b^2} - \frac{Gm_1m_2b}{(4d^2+b^2)^{\frac{3}{2}}} = \frac{Gm_1m_2}{b^2}(1-A) \tag{2}$$

where b is the distance between m_1 and m_2 and $A = \frac{b^3}{(4d^2+b^2)^{3/2}}$. This equation accounts for the attraction felt by each dumbbell mass to the close lead weight and the far lead weight. This force exerts a torque on the fiber given by

$$\tau_{gravity} = 2Fd = 2d\frac{Gm_1m_2}{b^2} \tag{3}$$

where d is half the length of the dumbbell. Combining equations 1 and 3, taking into account that the sum of all torques acting on the dumbbell must be zero, yields an expression for G:

$$G = \frac{\kappa\theta b^2}{2dm_1m_2} \tag{4}$$

A laser is shone on a mirror on the dumbbell and it reflects across the room on a wall. The laser-mirror system acted as a lever arm to amplify minuscule changes in θ to large position changes on the wall. Switching the arrangements of the weights resulted in a new equilibrium position. By measuring change in equilibrium position of the laser on the wall of ΔS, we calculated the change in equilibrium position

$$\theta = \frac{\Delta S}{4L} \tag{5}$$

The period of oscillation for a torsional balance is given by

$$T = 2\pi\sqrt{\frac{I}{\kappa}} = 2\pi\sqrt{\frac{2m_2(d^2+\frac{2}{5}r^2)}{\kappa}} \tag{6}$$

where T is the period of oscillation, I is the moment of inertia of the dumbbell ignoring the mirror and supporting arms, and r is the radius of the small weight. Solving this equation for κ and combining it with Equations 4 and 5 yields:

$$G = \frac{\pi^2\Delta S b^2(d^2+\frac{2}{5}r^2)}{T^2m_1^2Ld(1-A)} \tag{7}$$

Experiment

The dumbbell was offset from equilibrium with the weights in one position, and the experiment was allowed to run for several hours before the weights were shifted to the other position. Instead of manually recording data, a webcam, shown in Figure 1, was used to automate data acquisition. The computer was able to sample 3 times a second, and more importantly, was able to record data overnight. Previous experimentation indicated that far more than 2 periods were necessary for an accurate determination of equilibrium position. Python software analyzed each webcam image and identified any pixels within a set distance of the laser color, and it found the mean and standard deviation of the X coordinates of these pixels. A linear mapping exists between pixel position and the laser's position on the wall, determined through calibration. Any time the webcam was moved, the leftmost and rightmost wall coordinates in the webcam's field of view were entered into the software to compute this mapping. Thus, data runs taken without moving the webcam will have accurate relative position. The computer had far better time resolution than the webcam had position resolution, so time error was ignored. In addition, a USB thermometer took temperature readings of air directly around the apparatus every 5 seconds. It was located on the air-conditioning vent side of the apparatus. The thermometer had poor temperature resolution and was not calibrated, but for the purposes of this experiment, only relative temperature changes were important. The software Hid TEMPer was used for temperature data acquisition.

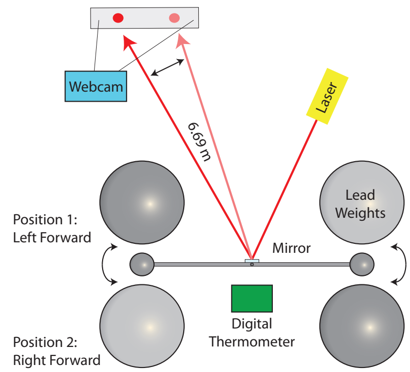

Figure 1: **Experiment Setup** The Cavendish experiment uses the gravitational attraction between a dumbbell and 2 lead weights to determine the gravitational constant. A laser reflects off a mirror on the dumbbell to amplify small changes in angular position. A webcam capture data efficiently and accurately.

Results

The raw data, the laser's position and the apparatus' temperature over time are shown in Figure 2. The initial os-

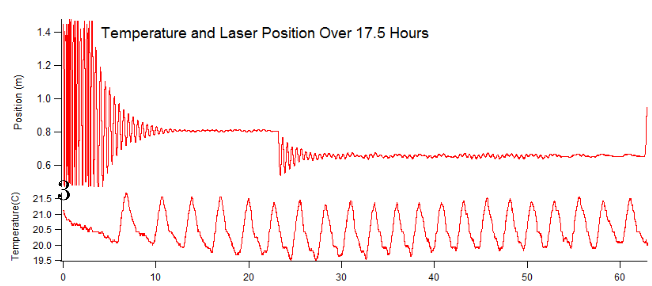

Figure 2: **Long-term Behavior** After some initial erroneous data, our data starts

cillations are erroneous, resulting from the dumbbell moving with so much energy as to bounce against the walls of the apparatus container instead of behaving as a torsional balance. At around 4,000 seconds, the dumbbell dissipates enough energy to behave normally. The curve is sinusoidal since it's a torsional balance, and decays exponentially due to damping in the torsional fiber. At around 23,000 seconds, the weights were moved from position 1 to position 2, resulting in a shift in equilibrium position.

Several graphs were used to fit this data over different time ranges. Period data is more accurate where the sinusoid's amplitude is high, but equilibrium position data is more accurate when the amplitude is low. The fit used was an exponentially decaying sinusoid:

$$x(t) = x_0 + Ae^{-t/\tau} sin(2\pi t/T_1 + \theta_1) \quad (8)$$

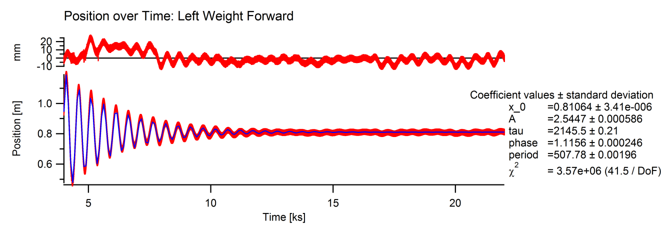

Figure 3: **Left Weight Forward**
Fitted curve to the data between 4,000 and 22,000 seconds. The damped sinusoid fits the data initially, but it does not explain why the oscillations don't die down.

where x_0 is the equilibrium position, A is the amplitude of the exponential decay envelope, τ is the decay parameter, T_1 is the period, and θ_1 is the phase. One of the graphs is shown in Figure 3. The damped sinusoid fits the data well at first, but it's insufficient to explain the continued oscillations after 12,500 seconds. The mean and standard deviation for the fit parameters were computed. The damping coefficient τ was 2170 ± 20 s, and the period T was 508.0 ± 0.2 s. For the left weight forward, the equilibrium position was 0.81621 ± 0.00002 m, and for the right weight forward, and the right position was 0.65629 ± 0.00005 m. These numbers were consistent over different time ranges. These parameters yield a value for G of $7.03 \cdot 10^{-11} \pm 0.15 \cdot 10^{-11}$ $\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$. The experimentally determined value of G was two standard deviations away from the accepted value. This could be caused by the temperature effect or a nonlinear mapping of pixels to X positions in the webcam.

Temperature's effect on the dumbbell's motion was determined through Fourier analysis. To remove the DC offset from the Fourier spectra, both the position and the temperature datasets were centered around zero. The position data was not taken at constant time intervals, so it was interpolated using the smoothing-spline method. Neither signal was intended to be periodic, but the FFT requires periodicity, so a Hanning window was applied to both datasets.

The results of the Fourier analysis of the temperature data is shown in Figure 4. The magnitude squared is plotted on a log plot for each Fourier coefficient. The only significant component of the temperature spectrum is the peak at 0.4 mHz. The frequency resolution of a Fourier transform is $\frac{1}{T_s}$ where T_s is the length of the signal, so the shown spectrum has a resolution of 0.02 mHz. The resulting period, 2500 ± 130 s is a reasonable timescale for an air-conditioning cycle.

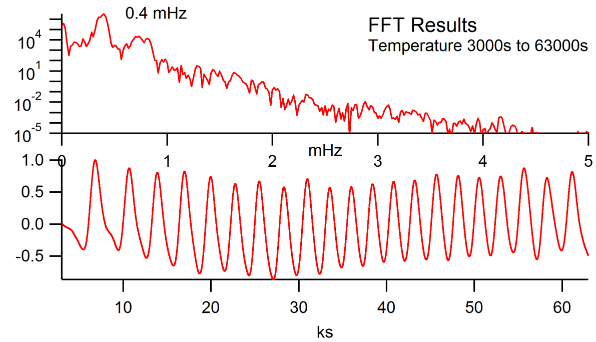

Figure 4: **Fourier Analysis of the Temperature** The top plot shows the spectrum of the temperature signal below. The range of the graph is the entire experiment. The peak at 0.4 mHz reflects the temperature's periodicity every 2500 seconds.

For the position data, shown in Figure 5, the peak at 1.88 mHz resulted from the characteristic period of the torsional dumbbell in the Cavendish apparatus. This position spectrum has a resolution of 0.04 mHz; thus, the 1.88 ± 0.04 mHz peak corresponds to a period of 532 ± 12 s, close enough to the period computed by fitting the position data to a damped sinusoid. Two other

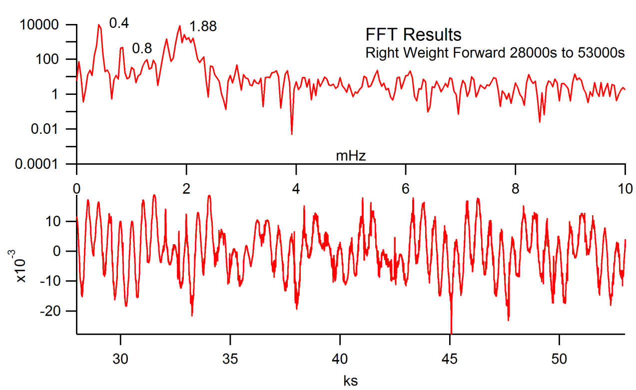

Figure 5: **Fourier Analysis of the Position** The magnitude squared of each Fourier coefficient is plotted on the top graph. The original signal is shown below. The 1.88 mHz component corresponds to a period of 532 seconds, close to our measured value. The other peaks at 0.4 and 0.8 mHz are a result of temperature fluctuations.

peaks occurred at 0.4 ± 0.04 mHz and 0.8 ± 0.04 mHz; the first is at the same frequency observed in the temperature spectrum, and the second is its harmonic. Similar frequency components were found in the data from the left weight forward as well.

The shared frequency component is strong evidence that temperature fluctuations manifested themselves in the dumbbell's movement. The temperature's effect does not appear to be due to metal expansion. The dumbbell's moment of inertia would only change by 0.005% over the range of temperatures observed. The torsional spring constant is even less effected by temperature. When the temperature increases, the increased radius of the beryllium copper wire is mostly canceled by a decrease in the modulus of rigidity. The most likely explaination is that environmental temperature variations created temperature differences within the apparatus. One side of the apparatus faced the air conditioning vent, while the other faced a wall. This temperature difference caused air currents that affect the dumbbell.

The nature of the FFT does not allow for accurate determination of the period of air conditioning cycle. However, we can update our model of the system to include the temperature fluctuations. From Fourier analysis, that there should be a sinusoidal component with a period near 2500 ± 130 s. The results are shown in Figure 6. The model is an improvement, but it is far from perfect. An exponentially damped sinusoid cannot model this because it can't account for amplitude increases provided by temperature. An improved model would be

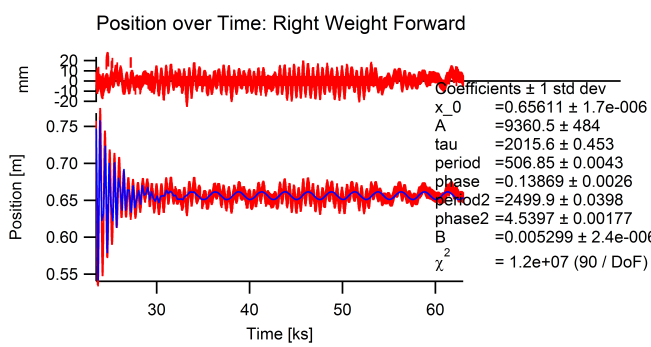

Figure 6: **Including Temperature in the Fit** Including temperature's additional affects yielded a graph that better fit the data, although this model doesn't account for the oscillations of period 500 s after the initial envelope decay. The fit shows that the period of the air conditioning cycles is 2499.9 ± 0.05 s, and their amplitude is roughly 5 mm.

$$x(t) = x_0 + (Ae^{-t/\tau} + B)sin(2\pi t/T_1 + \theta_1) + Csin(2\pi t/T_2 + \theta_2) \tag{9}$$

where C is the amplitude of the variations caused by temperature, T_2 and θ_2 are their period and phase, and B is minimum oscillation in the exponential decay envelope. This appears to coincide with observation: the oscillations initially decay exponentially, but they reach an equilibrium level where energy lost to damping equals the energy gained by temperature differences. However, this curve does not fit the data well because the period of these low-amplitude oscillations varies much more than it does for higher amplitude motion.

While the model used in Figure 6 ignores the 2 mHz components at high t, it does reveal information in its fit of the 0.4 mHz temperature oscillations. It pegs their period at 2499.9 ± 0.05 s, which makes sense if the cooling system is on a timer. The amplitude of the temperature oscillations is 5.300 ± 0.001 mm, large enough to be nonnegligable overnight. Someone recording only two periods of data for 1020 seconds could be anywhere in the 2500 second air conditioning cycle, so their equilibrium position values would only be accurate to 5 mm. All errors aside, this equilibrium position error would contribute to a 5% error in the calculation of G. It is unknown how the temperature cycle has similar effects during the daytime.

Conclusion

A value for the gravitational constant of $7.03 \cdot 10^{-11} \pm 0.15 \cdot 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$ was calculated. At least part of the error can be attributed to temperature - the benefit of an undisturbed lab for a long period of time comes with the cost of a devious air-conditioning system. Conclusive evidence was found linking temperature variations in the air around the Cavendish apparatus to the dumbbell's angular position. Chaotic behavior bars fully modeling the low-amplitude oscillations provoked by temperature differentials. New investigation into the relation between Cavendish temperature and torsional oscillation could be explored in several ways. More effective image processing would include a red filter and a blob detection algorithm. Inducing large heat variations with a heat gun should provoke an increased response that could be measured. Alternatively, enclosing the experiment in a metal box should reduce heat variations and improve the accuracy of the experiment.