





























summation. Thus, we must add back all the cases where there is a violation in *at least* two bins. For each pair of bins  $i, j = 1, \dots, n$  where  $i \neq j$ , we place  $r_i$  balls in  $i$  and  $r_j$  balls in  $j$  to guarantee violations in both. Then, we sort the remaining  $N - n - r_i - r_j$  balls into  $n$  bins using Lemma 2

$$\binom{(N - n - r_i - r_j) + n - 1}{n - 1} = \binom{N - r_i - r_j - 1}{n - 1}.$$

For all choices of  $i$  and  $j$ , the total number of cases where there is a violation in at least two bins is

$$\sum_{S \in T: |S|=2} \binom{N - \sum_{i \in S} r_i - 1}{n - 1}$$

which we must add back to our previous value. Thus, the new number of good cases is

$$\binom{N - 1}{n - 1} - \sum_{S \in T: |S|=1} \binom{N - \sum_{i \in S} r_i - 1}{n - 1} + \sum_{S \in T: |S|=2} \binom{N - \sum_{i \in S} r_i - 1}{n - 1}.$$

A pattern is beginning to take shape, but for the sake of demonstration, we will continue for one more iteration. Consider the cases where there is a size violation in 3 bins,  $i, j$ , and  $k$ . These cases would have been subtracted out 3 times when we considered all cases with violations in  $i, j$ , and  $k$  individually, but also added back in 3 times when we considered all the cases with violations in  $i$  and  $j, i$  and  $k$ , and  $j$  and  $k$ . These cancel out, and we have therefore not taken into account any of the cases where there are violations in

all three bins  $i, j$ , and  $k$ . In order to count these, we will proceed as before. To guarantee violations in all 3 bins, we place  $r_i$  balls in  $i$ ,  $r_j$  balls in  $j$ , and  $r_k$  balls in  $k$ , leaving us  $N - n - r_i - r_j - r_k$  balls to sort between  $n$  bins. Lemma 2 yields

$$\binom{(N - n - r_i - r_j - r_k) + n - 1}{n - 1} = \binom{N - r_i - r_j - r_k - 1}{n - 1}.$$

For all choices of  $i, j$ , and  $k$ , the total number of cases where there is a violation in at least three bins is

$$\sum_{S \in T: |S|=3} \binom{N - \sum_{i \in S} r_i - 1}{n - 1}$$

which we subtract from the total number of possible cases so far

$$\binom{N - 1}{n - 1} - \sum_{S \in T: |S|=1} \binom{N - \sum_{i \in S} r_i - 1}{n - 1} + \sum_{S \in T: |S|=2} \binom{N - \sum_{i \in S} r_i - 1}{n - 1} - \sum_{S \in T: |S|=3} \binom{N - \sum_{i \in S} r_i - 1}{n - 1}.$$

We continue this pattern of adding and subtracting bad cases where there are violations in  $x$  bins for  $x = 1, \dots, n$ , which will yield us the final formula

$$\sum_{k=0}^n \left( \sum_{S \in T: |S|=k} (-1)^k \binom{N - \sum_{i \in S} r_i - 1}{n - 1} \right).$$

□